cse 16 6-29-22

more logical identities :

Domination law

$$p \vee T = T$$
$$p \wedge F = F$$

Identity law

$$p \wedge T = p$$
$$p \vee F = p$$

Idempotent law

$$p \vee p = p$$
$$p \wedge p = p$$

Double negation law

$$\sim(\sim p) = p$$

Absorption law

$$p \vee (p \wedge q) = p$$

$$p \wedge (p \vee q) = p$$

Negation law

$$p \vee (\sim p) = T$$

$$p \wedge (\sim p) = F$$

Exercise

Prove all of these using truth tables.

Ex. show $(p \wedge q) \to (p \to q)$ is a tautology

$$(p \wedge q) \to (p \to q) = \sim(p \wedge q) \vee (\sim p \vee q) \quad \text{(implication)}$$

$$= (\sim p \vee \sim q) \vee (\sim p \vee q) \quad \text{(de Morg.)}$$

$$= (\sim p \vee \sim p) \vee (\sim q \vee q) \quad \text{(comm \& assoc)}$$

$$= \sim p \vee T \quad \text{(Idem, negation)}$$

$$= T \quad \text{(Domination)}$$

Exercise.

show $\sim(p \to q) \to p$ is a tautology

Exercise:

Determine whether

$$(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p)$$

is satisfiable.

set theory & logic

let A, B, C be subsets $\mathcal{U}$.

- $A \cap B = \{x \in \mathcal{U} \mid (x \in A) \wedge (x \in B)\}$
- $A \cup B = \{x \in \mathcal{U} \mid (x \in A) \vee (x \in B)\}$
- $A \oplus B = \{x \in \mathcal{U} \mid (x \in A) \oplus (x \in B)\}$
- $\overline{A} = \{x \in \mathcal{U} \mid \sim(x \in A)\}$

  $= \{x \in \mathcal{U} \mid x \notin A\}$
- $\phi = \{x \in \mathcal{U} \mid F\}$
- $\mathcal{U} = \{x \in \mathcal{U} \mid T\}$

analogy

sets		logic
set	$\longleftrightarrow$	statement
$\cap$	$\longleftrightarrow$	$\wedge$
$\cup$	$\longleftrightarrow$	$\vee$
$\oplus$	$\longleftrightarrow$	$\oplus$
(complement) $^{-}$	$\longleftrightarrow$	(negation) $\sim$
$\emptyset$	$\longleftrightarrow$	F
$\mathcal{U}$	$\longleftrightarrow$	T

Ex. DeMorgans laws for sets

① $\overline{A \cup B} = \overline{A} \cap \overline{B}$ ✓

② $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof of ①

$$\overline{A \cup B} = \{x \in U \mid \sim(x \in A \cup B)\}$$

$$= \{x \in U \mid \sim((x \in A) \vee (x \in B))\}$$

$$= \{x \in U \mid \sim(x \in A) \wedge \sim(x \in B)\}$$

$$= \{x \in U \mid x \in \overline{A} \wedge x \in \overline{B}\}$$

$$= \overline{A} \cap \overline{B}$$

Exercise: Prove ② the same way.

Every logical identity has an analog in set theory.

DeMorgan's laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

commutative laws

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Associative Laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

Domination laws

$$A \cup U = U$$
$$A \cap \phi = \phi$$

Identity laws

$$A \cap U = A$$
$$A \cup \phi = A$$

Idempotent laws

$$A \cup A = A$$
$$A \cap A = A$$

Double complement

$$\overline{\overline{A}} = A$$

Absorption laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \phi$$

Exercise:

Prove all of these set identities by reducing them to the corresponding logical identity.

Boolean Algebra

is the abstract algebraic system that generalizes

- propositional logic
- set theory
- circuit theory.

Thm If two expressions are equal, so are their duals.

(2.7) Quantifiers

The logic we've discussed so far is called

- Propositional logic

or
- 0^{th} order logic

1^{st} order logic

is an enlarged system that includes propositional functions (open sentences) and two new operators called <u>Quantifiers</u>

Recall a propositional function (predicate or open sentence) is a sentence $P(x)$ that becomes a stmt. when a value for x is specified.

Ex. $P(x) = (x < 7)$

$P(2) = (2 < 7)$ T

$P(10) = (10 < 7)$ F

Ex. $P(x) =$ 'x is a sunny day'

$P(\text{today}) =$ 'today is a sunny day'

Ex. $Q(x,y) = (x+y=10)$

$Q(3,7) = (3+7=10)$ T

$Q(3,8) = (3+8=10)$ F

The Universe of discourse or domain of a prop. fcn. is the set of values that can be substituted for its variable(s).

write: $\mathcal{U}$

also: $\mathcal{U}_x, \mathcal{U}_y$

Ex. $P(x,y) =$ 'day x is y degrees'

$\mathcal{U}_x = \{\text{days}\}, \quad \mathcal{U}_y = \mathbb{R}$

Quantification Operators

- Universal Quantifier: $\forall$ (for all)
- existential Quantifier: $\exists$ (there exists, for some)

Defn

- $\forall x \in \mathcal{U} : P(x)$

  $\forall x\, P(x)$ (if $\mathcal{U}$ is understood)

  "for all $x \in \mathcal{U}$, $P(x)$ is true"

- $\exists x \in \mathcal{U} : P(x)$

  $\exists x\, P(x)$

  "there exists an $x \in \mathcal{U}$ such that $P(x)$ is true."

  "there exists at least one $x \in \mathcal{U}$ s.t. $P(x)$"

  "for some $x \in \mathcal{U}$, $P(x)$"

Ex. $P(x) = (x^2 \geq 0)$ $\quad \mathcal{U} = \mathbb{R}$.

$\forall_x P(x) =$ "for all real numbers x, $x^2 \geq 0$"

= "the square of any real number is non-negative"

Ex. $Q(x) = (x < 50)$ $\quad \mathcal{U} = \mathbb{R}$

$\forall x\, Q(x) =$ "every real number is less than 50."

$\exists x\, Q(x) =$ "some real number is less than 50"

= "there is at least one real number that is less than 50"

Ex. $R(x) = (x^2 < 0)$ $\quad \mathcal{U} = \mathbb{R}$

$\exists x\, R(x) =$ "there is a real number whose square is negative"

Ex. $P(x,y) = (y^2 = x)$ $\quad \mathcal{U}_x = \mathcal{U}_y = \mathbb{R}$

$\forall x\, \exists y\, P(x,y) =$ "for every real x, there exists a real y such that $y^2 = x$."

$=$ "every real number has a real square root"

Ex. $R(x,y,z) = (x \cdot y = z)$

$$U_x = U_y = U_z = \mathbb{Q}$$

$$\forall x \, \forall y \, \exists z : R(x,y,z)$$

= "for any rational numbers x and y, there exists a rational number z such that $x \cdot y = z$."

= "the product of any two rational numbers is a rational number"

more .....