CSE 16 6-22-22

Continue . . . .

Ex.

- $\mathbb{P}(\emptyset) = \{\emptyset\}$ $\quad 2^0 = 1$

- $\mathbb{P}(\mathbb{P}(\emptyset)) = \mathbb{P}(\{\emptyset\})$

  $= \{\emptyset, \{\emptyset\}\}$ $\quad 2^1 = 2$

$\mathbb{P}(\{1\}) = \{\emptyset, \{1\}\}$

- $\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset))) = \mathbb{P}(\{\emptyset, \{\emptyset\}\})$

  $= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ $\quad 2^2 = 4$

$\mathbb{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Exercise

find an expression for

$$\Big|\underbrace{\mathbb{P}(\mathbb{P}(\mathbb{P}(\ldots\mathbb{P}}_{\#\,\mathbb{P}\text{'s} \,=\, n}(\phi)\ldots)))\Big|$$

(1.5) set operations

Defn

let A, B be sets.

- union : $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- difference : $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
$$= \{x \in A \mid x \notin B\}$$

Ex $A = \{1,2,3,4\}$, $B = \{3,4,5\}$

$$A \cup B = \{1, 2, 3, 4, 5\} = B \cup A$$

$$A \cap B = \{3,4\} = A \cap B$$

$$A - B = \{1,2\}$$

$$B - A = \{5\}$$

observe: (exercise)

$A - B \neq B - A$ unless $A = B$, in which case $A - B = \emptyset$.

Defn

We say A, B are *disjoint* iff $A \cap B = \emptyset$.

observe:

- for any A, B

$$(A - B) \cap B = \phi$$

- also

$$A - B \subseteq A$$

(1.6) complements

Suppose all sets under discussion are subsets of some universal set $\mathcal{U}$.

Defn

The complement of A (relative to $\mathcal{U}$) is

$$\begin{aligned} \overline{A} &= \mathcal{U} - A \\ &= \{x \in \mathcal{U} \mid x \notin A\} \end{aligned}$$

Ex. let $\mathcal{U} = \mathbb{N} = \{1, 2, 3, \dots\}$

let $A = \{x \in \mathcal{U} \mid x \geq 8\}$

$= \{8, 9, 10, \dots\}$

then

$$\overline{A} = \mathcal{U} - A$$

$$= \{x \in \mathcal{U} \mid x < 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 7\}.$$

Ex. $\mathcal{U} = \mathbb{R}$, $A = \{x \in \mathcal{U} \mid x \geq 8\}$

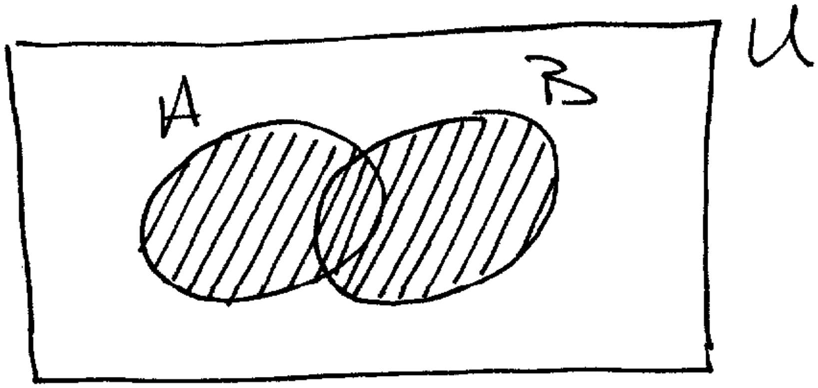
$= [8, \infty)$

then

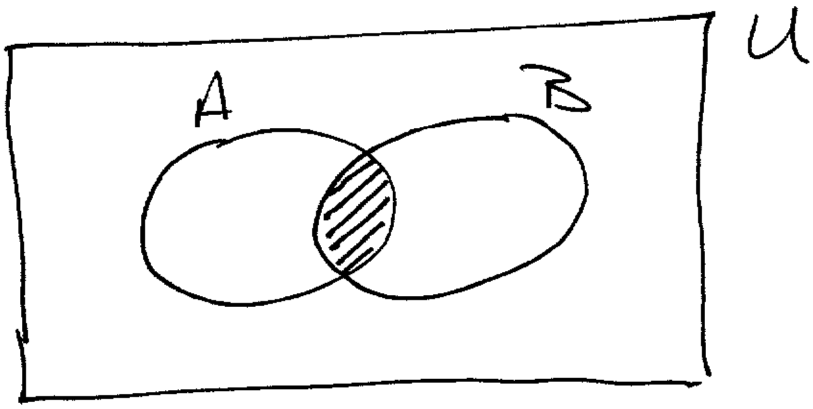
$$\overline{A} = \mathcal{U} - A = \{x \in \mathcal{U} \mid x < 8\}$$

$$= (-\infty, 8)$$

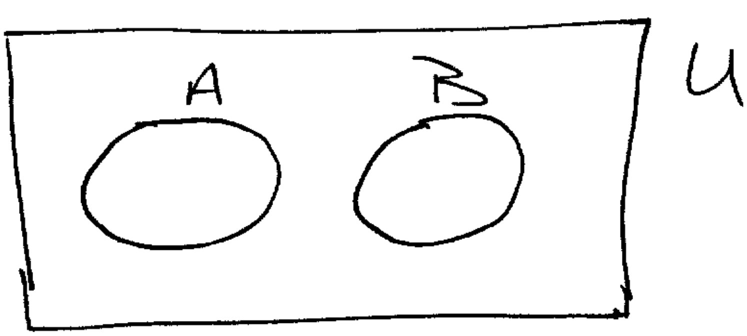
(1.7) Venn diagrams



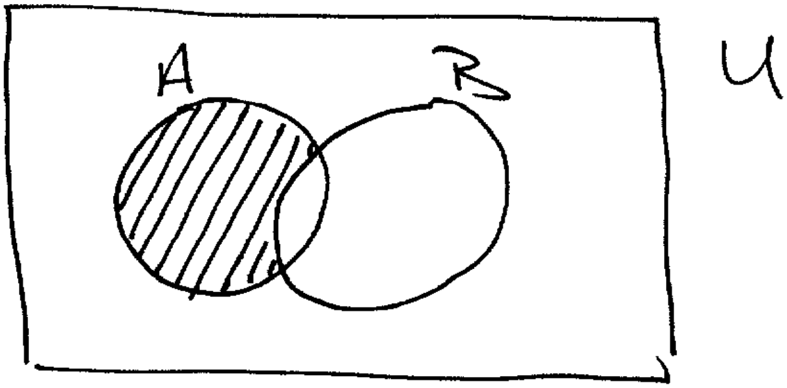
$A \cup B$



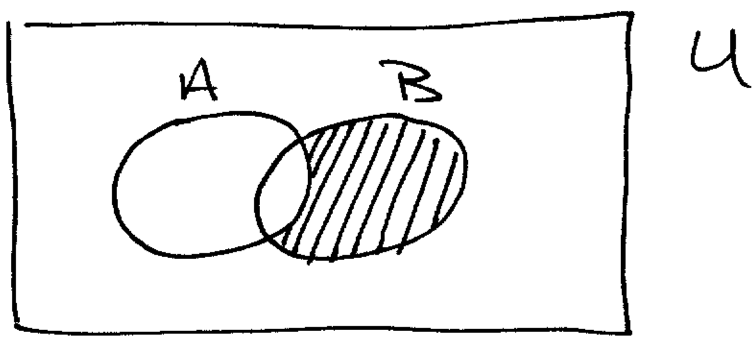
$A \cap B$



$A \cap B = \phi$



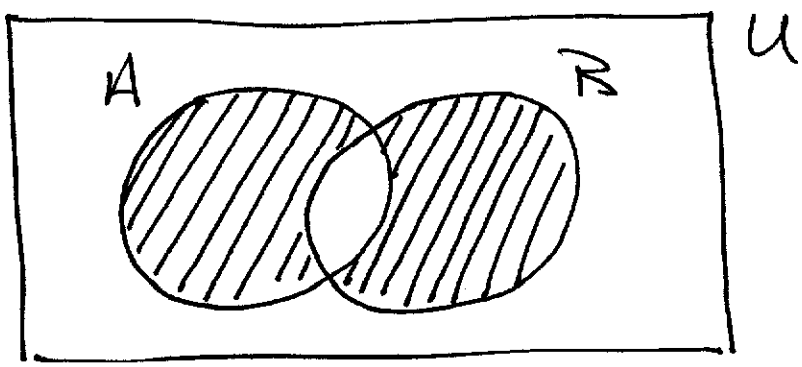
$A - B$



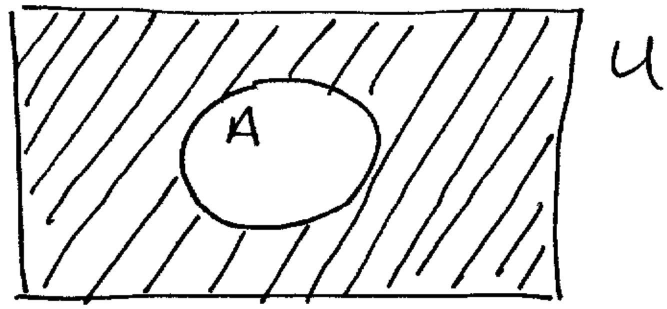
$B - A$

Defn symmetric difference

$$A \oplus B = (A - B) \cup (B - A)$$



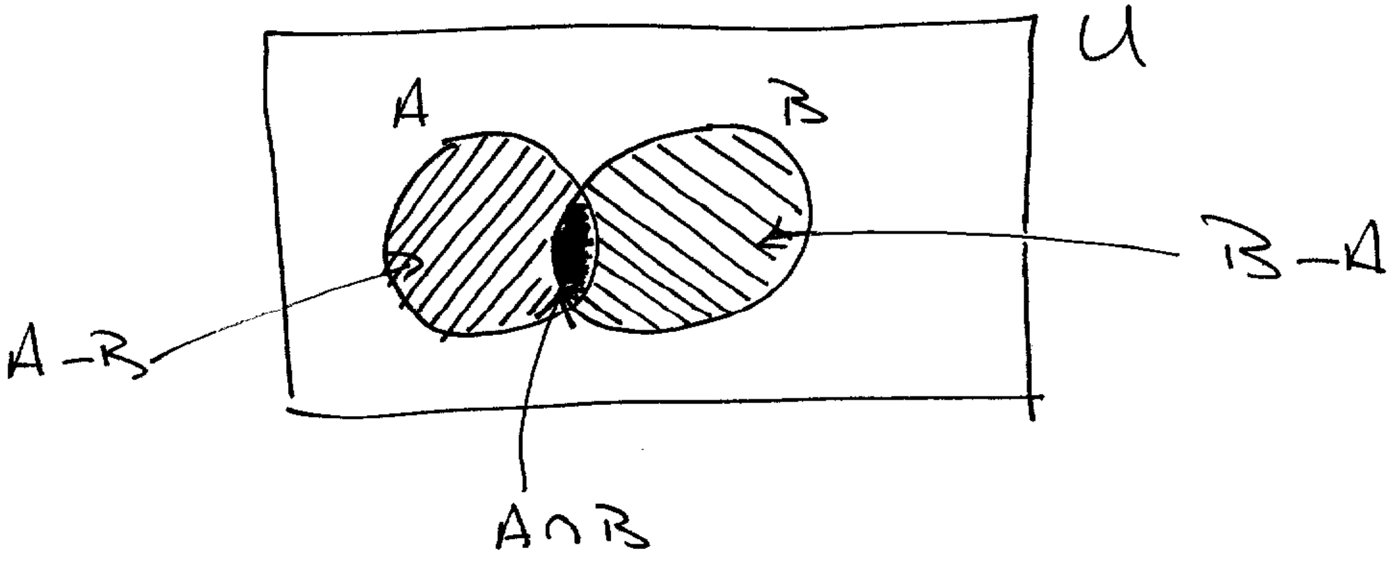
Observe: $A \oplus B = (A \cup B) - (A \cap B)$



$\bar{A} = U - A$

<u>Observe</u>:

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

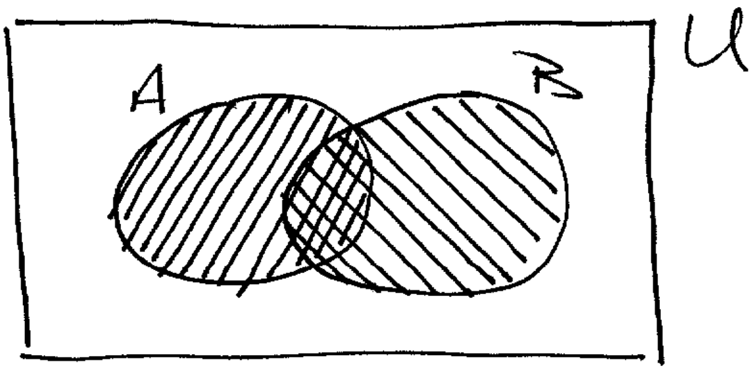


Theorem

if A, B are finite sets, then so are $A \cup B$ and $A \cap B$. Also

$$|A \cup B| = |A| + |B| - |A \cap B|$$

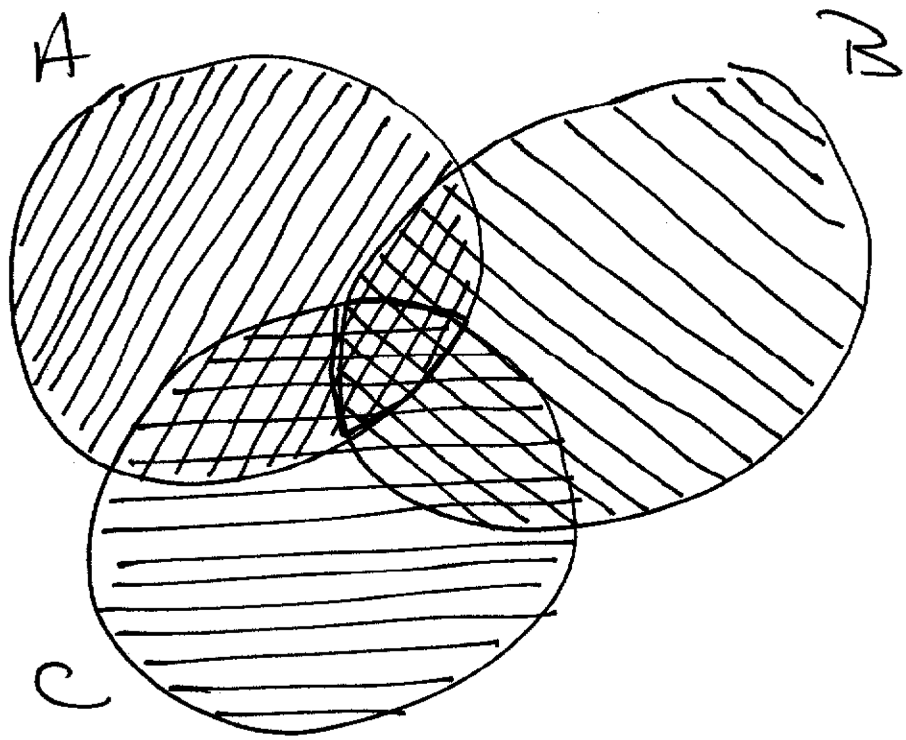
why?



This is a special case of a counting principle: PIE

Principle of Inclusion - Exclusion.

3-set version : A, B, C finite



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Exercise

- write 4-set version
- draw picture

(1.8) Indexed Sets

Let $A_1, A_2, A_3, \ldots\ldots$ be a (possibly infinite) list of sets.

<u>notation</u>

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$

$$= \{x \mid x \in A_i \text{ for } \underset{\text{(some)}}{\underline{\text{at least one}}} \ 1 \le i \le n\}$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots\cdots$$

$$= \{x \mid x \in A_i \text{ for } \underset{\text{(some)}}{\underline{\text{at least one}}} \ i \ge 1\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots\cdots \cap A_n$$

$$= \{x \mid x \in A_i \text{ for } \underline{\text{all}} \ 1 \le i \le n\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots\cdots$$

$$= \{x \mid x \in A_i \text{ for } \underline{\text{all}} \quad i \geq 1\}$$

we can pick out particular indices

<u>Ex</u>

$$\bigcup_{i \in \{2,7,11\}} A_i = A_2 \cup A_7 \cup A_{11}$$

likewise for $\cap$.

In general suppose we have a set A_α for $\alpha \in I$, where I is a (finite or infinet) set, called the <u>index set</u>

Then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for } \underline{\text{some}}\ \alpha \in I\}$$

and

$$\bigcap_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for } \underline{\text{all}}\ \alpha \in I\}$$

<u>Ex.</u> Suppose $J \neq \phi$ and $J \subseteq I$.

Does it follow that

$$(*) \qquad \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha \qquad ?$$

for instance: $J = \{1, 2\}$, $I = \{1, 2, 3\}$

(*) becomes

$$A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3$$

Proof of $(*)$

Suppose $x \in \bigcup_{\alpha \in J} A_\alpha$. Then for some $\beta \in J$ we have $x \in A_\beta$.

Since $J \subseteq I$, we also have $\beta \in I$.

$\therefore$ $x \in A_\beta$ for some $\beta \in I$.

$\therefore$ $x \in \bigcup_{\alpha \in I} A_\alpha$

It follows that:

$$\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$$

$\blacksquare$

outline of a Proof

hypothesis : $\phi \neq J \subseteq I$

hypothesis

statement

⋮

statement

Conclusion : $\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$

each statement follows from a previous stmt. by some

(1) definition

(2) inference rule

(3) previously proved fact

what do we make of expressions

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{and} \quad \bigcap_{\alpha \in I} A_\alpha$$

when $I = \phi$.

<u>Answer</u> if $I = \phi$

- $$\bigcup_{\alpha \in I} A_\alpha = \phi.$$

  so, could have left out hypothesis that $I \neq \phi$ in last example.

- requires a universal set $\mathcal{U}$

$$\bigcap_{\alpha \in I} A_\alpha = \mathcal{U}$$

if $I = \phi$.

(1.9) Number Systems

Recall that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

has special properties