CSE 16
Summer 2021
Quiz 4 **Solutions**

1. (25 Points) Prove that $\sqrt{7}$ is irrational. (Hint: you may use without proof that $7|n^2 \Rightarrow 7|n$ for any integer n.)

   Proof: (contradiction)
   Assume, to get a contradiction, that $\sqrt{7}$ is rational. Then $\sqrt{7} = a/b$ for some $a, b \in \mathbb{Z}$. If a and b have any common factors, we may cancel them. We can therefore assume, without loss of generality, that a and b have no common factors. In particular, ***a* and *b* <u>are not</u> both divisible by 7**. We have

$$
\begin{array}{lll}
 & 7 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} & \\
\therefore & 7b^2 = a^2 & \\
\therefore & 7|a^2 & \\
\therefore & \mathbf{7|a} & \text{(by the hint)} \\
\therefore & a = 7k & \\
\therefore & 7b^2 = (7k)^2 = 49k^2 & \\
\therefore & b^2 = 7k^2 & \\
\therefore & 7|b^2 & \\
\therefore & \mathbf{7|b} & \text{(by the hint)}
\end{array}
$$

   We have shown that ***a* and *b* <u>are</u> both divisible by 7**, contradicting our earlier remark. Therefore our assumption was false, and $\sqrt{7}$ is irrational. ■

2. (25 Points) Let $n, m, k \in \mathbb{Z}$, where $0 \le k \le m \le n$. Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$. You may use an algebraic or a combinatorial proof.

   Algebraic Proof:

$$
\begin{aligned}
\binom{n}{m}\binom{m}{k} &= \frac{n!}{m! \cdot (n-m)!} \cdot \frac{m!}{k! \cdot (m-k)!} \\
&= \frac{n!}{\cancel{m!} \cdot (n-m)!} \cdot \frac{\cancel{m!}}{k! \cdot (m-k)!} \cdot \frac{(n-k)!}{(n-k)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \cdot \frac{(n-k)!}{(m-k)! \cdot (n-m)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \cdot \frac{(n-k)!}{(m-k)! \cdot ((n-k)-(m-k))!} \\
&= \binom{n}{k}\binom{n-k}{m-k}.
\end{aligned}
$$

■

Combinatorial Proof:
Let S be a set with $|S| = n$. We count the number of ordered pairs (A, B) such that $A \subseteq B \subseteq S$ with $|A| = k$ and $|B| = m$, and we count them in two different ways.

I. First choose $B \subseteq S$ with $|B| = m$, then choose $A \subseteq B$ with $|A| = k$. The first step can be done in $\binom{n}{m}$ ways, and the second in $\binom{m}{k}$ ways. By the multiplication principle, the number of ways of making these choices in succession is $\binom{n}{m}\binom{m}{k}$, which is the left hand side.

II. First choose $A \subseteq S$ with $|A| = k$, then choose $C \subseteq S - A$ with $|C| = m - k$. When this is done set $B = A \cup C$, so $A \subseteq B \subseteq S$ with $|A| = k$ and $|B| = k + (m - k) = m$. The first choice can be made in $\binom{n}{k}$ ways, and the second in $\binom{n-k}{m-k}$ ways. By the multiplication principle, the number of ways of making these choices in succession is $\binom{n}{k}\binom{n-k}{m-k}$, which is the right hand side.

Since the left and right sides of the above formula count the very same set of objects, they must be equal, hence $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$. ■