CSE 16
Homework Assignment 7
Solutions to Selected Problems

<u>Chapter 10 (page 196): 26</u>

Prove (by induction) that $\forall n \geq 1$: $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, where F_n is the n^{th} Fibonacci number.

Proof:

Let $P(n)$ be the sentence $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We use weak induction form IIa.

I. $P(1)$ says $\sum_{k=1}^{1} F_k^2 = F_1 F_{1+1}$, i.e. $F_1^2 = F_1 F_2$, i.e. $1^2 = 1 \cdot 1$, or $1 = 1$, which is true.

II. Let $n \geq 1$. Assume $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We must show $\sum_{k=1}^{n+1} F_k^2 = F_{n+1} F_{n+2}$. We have

$$\begin{aligned}
\sum_{k=1}^{n+1} F_k^2 &= \left(\sum_{k=1}^{n} F_k^2\right) + F_{n+1}^2 \\
&= (F_n F_{n+1}) + F_{n+1}^2 && \text{by the induction hypothesis} \\
&= F_{n+1}(F_n + F_{n+1}) \\
&= F_{n+1} F_{n+2} && \text{by the Fibonacci recurrence}
\end{aligned}$$

It follows that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ for all $n \geq 1$ by the 1^{st} PMI. ■

Chapter 10 (page 196): 32

Prove that the number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}. For example, for $n = 2$ there are three such numbers (00, 01 and 10), and $3 = F_{2+2} = F_4$. Also for $n = 3$ there are five such numbers (000, 001, 010, 100, 101), and $5 = F_{3+2} = F_5$.

Proof:

Let B_n denote the number of bit strings of length n that contain no consecutive 1's. These can also be described as bit strings of length n that do not contain the substring 11. We will prove that $B_n = F_{n+2}$ for all $n \geq 1$. We use strong induction form IId with two base cases.

I. If $n = 1$, then there are two such strings (1 and 0). Therefore $B_1 = 2 = F_3 = F_{1+2}$.

If $n = 2$, then (as noted in the problem statement) there are three such strings (00, 01 and 10). Therefore $B_2 = 3 = F_4 = F_{2+2}$. The two base cases are now satisfied.

II. Let $n > 2$ be arbitrary, and assume for all k in the range $1 \leq k < n$ that $B_k = F_{k+2}$. We must show that $B_n = F_{n+2}$.

Let s be a bit string of length n that does not contain the substring 11. The rightmost bit in s is necessarily 0 or 1, giving us two (mutually exclusive) cases for how s must end.

(1) The string s ends in 0:

$$s = \underbrace{x\,x\,x\,\cdots\,x}_{n-1}\,0$$

In this case, the preceding $n - 1$ bits cannot contain the substring 11, and hence there are B_{n-1} ways to choose s.

(2) The string s ends in 1, so actually ends in 01 (since it cannot contain 11):

$$s = \underbrace{x\,x\,x\,\cdots\,x}_{n-2}\,0\,1$$

In this case, the preceding $n - 2$ bits cannot contain the substring 11, and therefore there are B_{n-2} ways to choose s.

By the addition principle, the number of choices for s is $B_n = B_{n-1} + B_{n-2}$. The Induction hypothesis (cases $k = n - 1$ and $k = n - 2$) gives us $B_{n-2} = F_{(n-2)+2} = F_n$ and $B_{n-1} = F_{(n-1)+2} = F_{n+1}$. Putting these facts together, and using the Fibonacci recurrence formula, we have

$$B_n = B_{n-1} + B_{n-2} = F_n + F_{n+1} = F_{n+2},$$

as required. The result now holds for all $n \geq 1$ by the 2nd PMI. ■

<u>(11.1) 6</u>

Congruence modulo 5 is a relation on the set $A = \mathbb{Z}$. In this relation xRy means that $x \equiv y \pmod 5$. Write out the set R in set-builder notation.

Solution:

Since $x \equiv y \pmod 5$ is defined to mean $5 \mid (x - y)$, which in turn means $x - y = 5k$ for some $k \in \mathbb{Z}$, we have

$$\begin{aligned} R &= \{ (x, y) \in \mathbb{Z} \times \mathbb{Z} : \ 5 \mid (x - y) \} \\ &= \{ (x, y) \in \mathbb{Z} \times \mathbb{Z} : \ x - y = 5k \text{ for some } k \in \mathbb{Z} \} \end{aligned}$$

(Note to grader: either form above is acceptable.) ■

<u>(11.2) 14</u>

Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.

Proof:

Let x be an arbitrary element of A. We are given that aRx, and therefore xRa since R is symmetric. But now the transitivity of R says $xRa \wedge aRx \Rightarrow xRx$. Since x was arbitrary, we have xRx for all $x \in A$, showing that R is reflexive. ■