CSE 16
Homework Assignment 6
Solutions to Selected Problems

Chapter 6 (page 144): 16
If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Proof:
For any real numbers x and y we have $(x - y)^2 \geq 0$. Therefore $x^2 - 2xy + y^2 \geq 0$, and $x^2 + y^2 \geq 2xy$. Now given $a, b \in \mathbb{R}^+$, set $x = \sqrt{a}$ and $y = \sqrt{b}$. Then $a + b = x^2 + y^2 \geq 2xy = 2\sqrt{a}\sqrt{b} = 2\sqrt{ab}$. ■

Chapter 7 (page 155): 30
Let $a, b, p \in \mathbb{Z}$, and suppose p is prime. Prove that if $p|ab$, then either $p|a$ or $p|b$. (Hint: use the proposition on page 152, which says $\gcd(u, v) = ux + vy$ for some $x, y \in \mathbb{Z}$.)

Proof:
Assume $p|ab$, whence $ab = pk$ for some $k \in \mathbb{Z}$. Either p divides a, or it does not.

Case 1: $p|a$
In this case we are done.

Case 2: $p \nmid a$
Since p is prime, the divisors of p are $\{1, p\}$. The only common divisor of p and a is therefore 1, hence $\gcd(p, a) = 1$. By the hint (a fact that was proved in class), we have $px + ay = 1$, for some $x, y \in \mathbb{Z}$. Multiplying this last equation by b, we have

$$b = b(px + ay) = pbx + aby = pbx + pky = p(bx + ky),$$

showing that $p|b$.

Since at least one of these cases must hold, we have either $p|a$ or $p|b$, as required. ■

<u>Chapter 10 (page 195): 8</u>

Prove (by induction) that $\forall n \geq 1$:

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof:

I. If $n = 1$, the above statement is $1/(1+1)! = 1 - 1/(1+1)!$, i.e. $1/2 = 1/2$, which is true.

II. Let $n \geq 1$ be chosen arbitrarily. Assume, for this n, that

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}.$$

We must show

$$\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+2)!}.$$

Observe that

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{k}{(k+1)!} &= \left(\sum_{k=1}^{n} \frac{k}{(k+1)!}\right) + \frac{n+1}{(n+2)!} \\
&= \left(1 - \frac{1}{(n+1)!}\right) + \frac{n+1}{(n+2)!} \qquad \text{(by the induction hypothesis)} \\
&= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!} \\
&= 1 - \left(\frac{(n+2)-(n+1)}{(n+2)!}\right) \\
&= 1 - \frac{1}{(n+2)!}.
\end{aligned}$$

The result follows for all $n \geq 1$ by the Principle of Mathematical Induction. ∎

Chapter 10 (page 195): 12

Prove (by induction) that $\forall n \geq 0$: $9|(4^{3n} + 8)$.

Proof:

I. If $n = 0$, then $4^{3n} + 8 = 4^0 + 8 = 1 + 8 = 9$, hence $9|(4^{3\cdot 0} + 8)$, establishing the base case.

II. Let $n \geq 0$ be chosen arbitrarily. Assume, for this n, that $9|(4^{3n} + 8)$. Therefore $4^{3n} + 8 = 9k$ for some $k \in \mathbb{Z}$. We must show that $9|(4^{3(n+1)} + 8)$. Observe

$$\begin{aligned}
4^{3(n+1)} + 8 &= 4^{3n+3} + 8 \\
&= 4^3 \cdot 4^{3n} + 8 \\
&= (63 + 1) \cdot 4^{3n} + 8 \\
&= 63 \cdot 4^{3n} + (4^{3n} + 8) \\
&= 9(7 \cdot 4^{3n}) + 9k \qquad \text{By the induction hypothesis} \\
&= 9(7 \cdot 4^{3n} + k),
\end{aligned}$$

showing that $9|(4^{3(n+1)} + 8)$ as required.

The result follows for all $n \geq 0$ by the Principle of Mathematical Induction. ■