

CSE 16
Homework Assignment 5
Solutions to Selected Problems

Chapter 4 (page 126): 12

If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.

Proof:

Since $x < 4$ we have $4 - x > 0$. Together with $0 < x$, this gives $x(4 - x) > 0$. Also

$$(x - 2)^2 \geq 0 \Rightarrow x^2 - 4x + 4 \geq 0$$

$$\Rightarrow 4 \geq 4x - x^2$$

$$\Rightarrow 4 \geq x(4 - x).$$

Now divide both sides by the positive number $x(4 - x)$ to obtain

$$\frac{4}{x(4 - x)} \geq 1.$$

As required. ■

Chapter 5 (page 136): 6

Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

Proof:

Assume $x \leq -1$. It follows that $x + 1 \leq 0$. Also $x - 1 \leq -1 - 1 = -2$. Since the product of three non-positive numbers is non-positive, we have

$$x(x - 1)(x + 1) \leq 0,$$

$$\therefore x(x^2 - 1) \leq 0,$$

$$\therefore x^3 - x \leq 0.$$

We've shown that $x \leq -1 \Rightarrow x^3 - x \leq 0$. By contraposition, $x^3 - x > 0 \Rightarrow x > -1$. ■

Chapter 5 (page 136): 18

If $a, b \in \mathbb{Z}$, then $(a + b)^3 \equiv a^3 + b^3 \pmod{3}$.

Proof:

Observe $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = (a^3 + b^3) + 3(a^2b + ab^2)$, so that

$$(a + b)^3 - (a^3 + b^3) = 3(a^2b + ab^2).$$

Thus $3 \mid ((a + b)^3 - (a^3 + b^3))$, whence $(a + b)^3 \equiv a^3 + b^3 \pmod{3}$. ■

Chapter 6 (page 144): 6

If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

Proof:

Assume, to get a contradiction, that $a^2 - 4b - 2 = 0$. Then $a^2 = 4b + 2 = 2(2b + 1)$. This shows $2 \mid a^2$, and by a result proved in class, we have $2 \mid a$. Therefore $a = 2k$ for some $k \in \mathbb{Z}$, and

$$\begin{aligned} (2k)^2 &= 2(2b + 1) \\ \therefore 4k^2 &= 2(2b + 1) \\ \therefore 2k^2 &= 2b + 1. \end{aligned}$$

But the left hand side of this equation is even, while the right hand side is odd, which is impossible. This contradiction shows our assumption was false, and therefore $a^2 - 4b - 2 \neq 0$, as claimed. ■