CSE 16
Homework Assignment 4
Solutions to Selected Problems

<u>**(3.7) 4c**</u>

How many 6-letter lists from the alphabet $\mathcal{S} = \{\mathrm{T, H, E, O, R, Y}\}$ (with repetition allowed) are there in which the sequence of letters (T, H, E) appears consecutively (i.e. in that order)?

Solution:

We define 4 sets of lists, based on the position of the sequence (T, H, E):

$$\begin{aligned}
A &= \{\ (\mathbf{T}, \mathbf{H}, \mathbf{E}, X, Y, Z)\ : X, Y, Z \in \mathcal{S}\ \},\\
B &= \{\ (X, \mathbf{T}, \mathbf{H}, \mathbf{E}, Y, Z)\ : X, Y, Z \in \mathcal{S}\ \},\\
C &= \{\ (X, Y, \mathbf{T}, \mathbf{H}, \mathbf{E}, Z)\ : X, Y, Z \in \mathcal{S}\ \},\\
D &= \{\ (X, Y, Z, \mathbf{T}, \mathbf{H}, \mathbf{E})\ : X, Y, Z \in \mathcal{S}\ \}.
\end{aligned}$$

The 4-set version of the Inclusion-Exclusion principle says

$$\begin{aligned}
|A \cup B \cup C \cup D| = \ & |A| + |B| + |C| + |D|\\
& -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|\\
& +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|\\
& -|A \cap B \cap C \cap D|
\end{aligned}$$

Notice however that because of conflicts in the list positions 2, 3, 4 and 5, all but one of the 2-set intersections is empty. The only non-empty 2-set intersection is $A \cap D = \{\ (\mathrm{T, H, E, T, H, E})\ \}$, containing exactly one list. As a consequence, all of the 3-set and 4-set intersections are empty as well. Observe also $|A| = |B| = |C| = |D| = 6^3 = 216$. It follows then that

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap D| = \mathbf{4 \cdot 216 - 1 = 863}.$$

■

<u>**(3.8) 8**</u>

How many lists (x, y, z) of three integers are there with $0 \leq x \leq y \leq z \leq 100$?

Solution:

Each list corresponds to a 3-element multiset $[x, y, z]$ based on the 101-element set $\{0, 1, 2, 3, \ldots \ldots, 100\}$. By a formula derived in class (p.3 of the notes from 7-14-21), the number of such lists is

$$\binom{3 + 101 - 1}{3} = \binom{103}{3} = \mathbf{176{,}851}.$$ ■

Alternate Solution:

Given a list (x, y, z) satisfying $0 \leq x \leq y \leq z \leq 100$, define new variables $a = x, b = y - x, c = z - y$ and $d = 100 - z$. Then $a + b + c + d = 100$ where $a \geq 0, b \geq 0, c \geq 0, d \geq 0$. Thus, the required lists correspond exactly to the non-negative integer solutions (a, b, c, d) to $a + b + c + d = 100$. By an example covered in class (p.5 of the notes from 7-14-21), these solutions correspond to the 100-element multisets based on a 4-element set. The number of such lists is therefore

$$\binom{100 + 4 - 1}{100} = \binom{103}{100} = \binom{103}{3} = \mathbf{176{,}851}$$ ■

(3.9) 2

You deal a pile of cards, face down, from a standard 52-card deck. What is the least number of cards the pile must have before you can be assured that it contains at least five cards of the same suit?

Solution:

Let n denote the number of cards in the pile. Since there are 4 suits in a standard deck, the pigeonhole principle (also called the *division principle* in the text), tells us to seek the least n for which

$$\left\lceil \frac{n}{4} \right\rceil = 5.$$

This is equivalent to the inequality $4 < n/4 \leq 5$, which is the same as $16 < n \leq 20$. The smallest such number is $\boldsymbol{n} = \mathbf{17}$. ∎

(3.10) 6

Show that $\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$.

Proof:

Let S be a set with $|S| = 3n$, and let A, B and C be subsets of S satisfying $A \cap B = A \cap C = B \cap C = \emptyset$, and $|A| = |B| = |C| = n$. The left hand side of the above formula, by its very definition, counts the number of 3-element subsets of S, i.e. the number of $\{x, y, z\} \subseteq S$, with x, y, z distinct. We count these same subsets in another way to obtain the right hand side.

Partition the 3-element subsets of S into 3 mutually exclusive classes.

I. Choose $\{x, y, z\} \subseteq A$: #ways $= \binom{n}{3}$
   Choose $\{x, y, z\} \subseteq B$: #ways $= \binom{n}{3}$
   Choose $\{x, y, z\} \subseteq C$: #ways $= \binom{n}{3}$
   Total number of sets of this type: $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3}$

II. Choose $x \in A$ and $\{y, z\} \subseteq B$: #ways $= n\binom{n}{2}$
   Choose $x \in A$ and $\{y, z\} \subseteq C$: #ways $= n\binom{n}{2}$
   Choose $x \in B$ and $\{y, z\} \subseteq A$: #ways $= n\binom{n}{2}$
   Choose $x \in B$ and $\{y, z\} \subseteq C$: #ways $= n\binom{n}{2}$
   Choose $x \in C$ and $\{y, z\} \subseteq A$: #ways $= n\binom{n}{2}$
   Choose $x \in C$ and $\{y, z\} \subseteq B$: #ways $= n\binom{n}{2}$
   Total number of sets of this type: $6n\binom{n}{2}$

III. Choose $x \in A$, $y \in B$ and $z \in C$: #ways $= n \cdot n \cdot n = n^3$

Since every 3-element subset $\{x, y, z\} \subseteq S$ falls into exactly one of the above classes, the addition principle gives the number of such subsets as $3\binom{n}{3} + 6n\binom{n}{2} + n^3$, which proves the formula. ∎