CSE 16
Homework Assignment 3
Solutions to Selected Problems

(3.3) 10

Consider the lists of length six made with the symbols $\{P, R, O, F, S\}$ where repetition is allowed. How many such lists can ge made if the list must end with S, and the symbol O is used more than once?

Solution:

Our list will have 6 positions $\{1,2,3,4,5,6\}$. Position 6 will be occupied by S, and positions $\{1,2,3,4,5\}$ will be chosen from $\{P, R, O, F, S\}$. At minimum, 2 of these must be O's. For instance:

$$\underset{1}{\underline{\text{O}}}\ \underset{2}{\underline{\text{O}}}\ \underset{3}{\underline{\ \ }}\ \underset{4}{\underline{\text{O}}}\ \underset{5}{\underline{\ \ }}\ \underset{6}{\underline{\text{S}}}\ .$$

The set of such lists can be partitioned into 4 subsets, based on the number of O's appearing in the list. This number, which we denote as n, can be 2, 3, 4 or 5.

(1) $n = 2$. Choose which 2 positions from $\{1,2,3,4,5\}$ are to be occupied by O's, then choose symbols for the remaining 3 positions from $\{P, R, F, S\}$ (where repetition is allowed). By the multiplication principle this can be done in $\binom{5}{2} \cdot 4 \cdot 4 \cdot 4 = 10 \cdot 64 = 640$ ways.

(2) $n = 3$. Choose which 3 positions from $\{1,2,3,4,5\}$ are to be occupied by O's, then choose symbols for the remaining 2 positions from $\{P, R, F, S\}$ (repetition allowed). By the multiplication principle this can be done in $\binom{5}{3} \cdot 4 \cdot 4 = 10 \cdot 16 = 160$ ways.

(3) $n = 4$. Choose which 4 positions from $\{1,2,3,4,5\}$ are to be occupied by O's, then choose a symbol for the remaining 1 position from $\{P, R, F, S\}$ (repetition allowed). By the multiplication principle this can be done in $\binom{5}{4} \cdot 4 = 5 \cdot 4 = 20$ ways.

(4) $n = 5$. Let all 5 positions from $\{1,2,3,4,5\}$ be occupied by O's. This can be done in just 1 way.

By the addition principle, there are $640 + 160 + 20 + 1 = \mathbf{821}$ such lists. ■

(3.4) 16

How many 4-permutations are there of the set $\{A, B, C, D, E, F\}$ if whenever A appears in the permutation, it is followed by E?

Solution:

This set of permutations partitions into two subsets: those containing A, and those not containing A. We deal with these two subsets separately, then use the addition principle.

(1) If the permutation does not contain A, then it is just a 4-permutation of the set $\{B, C, D, E, F\}$. The number of such permutations is $P(5,4) = 5 \cdot 4 \cdot 3 \cdot 2 = 120$.

(2) If the permutation does contain A, then since A must be followed by E, we can only place the A in positions $\{1, 2, 3\}$, as follows:

A E - -
- A E -
- - A E

After placing the E after the A, there are two positions left to be filled from the set $\{B, C, D, F\}$. This can be done in $P(4,2) = 4 \cdot 3 = 12$ ways. By the multiplication principle, the number of permutations containing A is $3 \cdot 12 = 36$.

By the addition principle, the total number of permutations is $120 + 36 = \mathbf{156}$. ■

(3.5) 4

Suppose the set B has the property that $|\{X \mid X \in \mathbb{P}(B) \text{ and } |X| = 6\}| = 28$. Determine $|B|$.

Solution:

Let $|B| = n$. It is required that B have exactly 28 subsets of cardinality 6, so we must solve the equation

$$\binom{n}{6} = 28$$

for n. Using the formula for binomial coefficients, we need to find the roots of a 6^{th} degree polynomial. It's much easier to just examine Pascal's triangle until we find the number 28 in the 6^{th} column. We find that this occurs in row 8:

$$\binom{8}{6} = 28,$$

and hence $|\boldsymbol{B}| = \mathbf{8}$. ■

(3.6) 10

Show that the formula $k\binom{n}{k} = n\binom{n-1}{k-1}$ is true for all integers n, k satisfying $0 \le k \le n$.

Solution:

The wording of the question does not specify if this is to be a combinatorial or an algebraic proof. We will accept either, and present both below.

Combinatorial Proof:

Suppose we have a group of n persons. In how many ways can we pick a committee consisting of k of these persons, with one member designated as the chair-person? We can solve this problem in two ways, as follows.

a. First pick a k-element subset of our group of n persons, then pick one of the k to be chair-person. By the multiplication principle these choices can be made in $k\binom{n}{k}$ ways, which is the left hand side of the equation.

But another procedure can also be used to count the number of committiees with chair-person.

b. First pick the chair-person from among the n people, then from the remaining $(n-1)$ persons, choose the other $(k-1)$ members of the committee. By the multiplication principle, this can be done in $n\binom{n-1}{k-1}$ ways, which is the right hand side of the equation.

Since both sides of the equation count the very same thing (the number of ways of picking a committee of size k with designated chair-person from a group of n persons), the two sides must be equal, i.e.

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

■

Algebraic Proof:

$$\begin{aligned}
k\binom{n}{k} &= k \cdot \frac{n!}{k!\,(n-k)!} \\
&= \frac{n!}{(k-1)!\,(n-k)!} \\
&= n \cdot \frac{(n-1)!}{(k-1)!\,(n-k)!} \\
&= n \cdot \frac{(n-1)!}{(k-1)!\,\big((n-1)-(k-1)\big)!} \\
&= n\binom{n-1}{k-1}
\end{aligned}$$

■