closed formula for F_n, the n^{th} Fibonacci number.

Define constants:

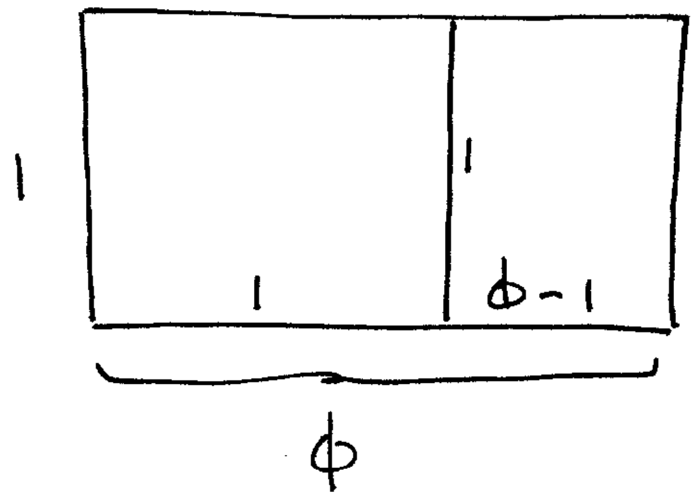
$$\phi_1 = \frac{1+\sqrt{5}}{2} = 1.236\ldots \Big\} \text{ Golden Ratio}$$

$$\phi_2 = \frac{1-\sqrt{5}}{2} = -0.618\ldots \Big\} \text{ Conjugate of Golden Ratio.}$$

Roots of

$$(*) \qquad x^2 - x - 1 = 0$$

what's 'golden' about ϕ_1 ?



$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

so
$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

$$\phi(\phi - 1) = 1$$

$$\phi^2 - \phi - 1 = 0$$

or
$$\boxed{\phi^2 = \phi + 1}$$

P(n)

<u>Theorem</u>

$$\forall n \geq 1 : \boxed{F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right)}$$

<u>Proof.</u> (strong ind. IId, 2 base cases)

I. <u>$n=1$</u>: $\text{RHS} = \frac{1}{\sqrt{5}}\left(\phi_1^{1} - \phi_2^{1}\right)$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right)$$

$$= \frac{1}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{2}\cancel{\sqrt{5}}}{\cancel{2}} = 1 = F_1 = \text{LHS}.$$

<u>$n=2$</u>: $\text{RHS} = \frac{1}{\sqrt{5}}\left(\phi_1^{2} - \phi_2^{2}\right)$

$$= \frac{1}{\sqrt{5}}\left((\phi_1 + \cancel{1}) - (\phi_2 + \cancel{1})\right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi_1 - \phi_2\right)$$

$$= 1 = F_2 = \text{LHS}$$

II d. $\forall n > 2 : \left(P(1) \wedge \cdots \wedge P(n-1) \right) \Rightarrow P(n)$

Let $n > 2$ be arbitrary. Assume for all k in the range $1 \le k < n$ that

$$F_k = \frac{1}{\sqrt{5}} \left(\phi_1^k - \phi_2^k \right).$$

we must show

$$F_n = \frac{1}{\sqrt{5}} \left(\phi_1^n - \phi_2^n \right)$$

so

$$RHS = \frac{1}{\sqrt{5}} \left(\phi_1^n - \phi_2^n \right)$$

$$= \frac{1}{\sqrt{5}} \left(\phi_1^2 \cdot \phi_1^{n-2} - \phi_2^2 \cdot \phi_2^{n-2} \right)$$

$$= \frac{1}{\sqrt{5}} \left((\phi_1 + 1) \phi_1^{n-2} - (\phi_2 + 1) \phi_2^{n-2} \right)$$

$$= \frac{1}{\sqrt{5}}\left((\phi_1^{n-1} + \phi_1^{n-2}) - (\phi_2^{n-1} + \phi_2^{n-2}) \right)$$

$$= \frac{1}{\sqrt{5}}\left((\phi_1^{n-1} - \phi_2^{n-1}) + (\phi_1^{n-2} - \phi_2^{n-2}) \right)$$

$$= \frac{1}{\sqrt{5}}(\phi_1^{n-1} - \phi_2^{n-1}) + \frac{1}{\sqrt{5}}(\phi_1^{n-2} - \phi_2^{n-2})$$

$$= F_{n-1} + F_{n-2}$$ $\left\{ \begin{array}{l} \text{by the ind.} \\ \text{hyp. with} \\ k = n-1,\ k = n-2 \end{array} \right.$

$$= F_n$$ $\left\{ \begin{array}{l} \text{by the} \\ \text{Fib. Recurrence} \end{array} \right.$

$$= LHS.$$

Result follows for all $n \geq 1$ by the 2^{nd} PMI. ∎

Exercises

- ch 10 #25 :

$$\forall n \geq 1 : \sum_{k=1}^{n} F_k = F_{n+2} - 1$$

- ch 10 #29 :

$$\forall n \geq 1 : \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

note: $\binom{a}{b} = 0$ when $b < 0$ or $a < b$.

- Prove $\forall n \geq 1 : 2 \mid (n^2 + n)$
- " $\forall n \geq 0 : 6 \mid (n^3 - n)$
- " $\forall n \geq 1 :$ n odd $\Rightarrow 8 \mid (n^2 - 1)$

- " $\forall n \geq 1 : 21 \mid (4^{n+1} + 5^{2n-1})$
- " $\forall n \geq 1 : 133 \mid (11^{n+1} + 12^{2n-1})$
- " $\forall n \geq 2 : F_{n+1} F_{n-1} - F_n^2 = (-1)^n$
- Show $\forall n \geq 1 : \gcd(F_{n+1}, F_n) = 1$.
- Determine the # of divisions needed in Euclidean Algorithm to compute $\gcd(F_{n+1}, F_n)$. Prove your answer by induction

chapter 11 : Relations

(11.1) General Relations

Defn

A relation from A to B (sets) is a subset:

$$R \subseteq \underbrace{A}_{\text{domain}} \times \underbrace{B}_{\text{co-domain}} .$$

notation

given $x \in A$, $y \in B$, write

$x R y$ to mean $(x, y) \in R$

and

$x \not{R} y$ " " $(x, y) \notin R$

more generally a relation on sets $A_1, A_2, \ldots, A_n$ is a subset

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n$$

(called an <u>n-ary relation</u>).

<u>Defn</u>

A <u>relation on A</u> is a relation from A to A, i.e.

$$R \subseteq A \times A$$

<u>Examples</u>: relations on $\mathbb{Z}$.

- $<$ is the set $\{(x,y) \mid x<y\} \subseteq \mathbb{Z}\times\mathbb{Z}$

  so $(1,2)\in <$ and $(2,1)\notin <$

- $\leq$ " " " $\{(x,y) \mid x\leq y\} \subseteq \mathbb{Z}\times\mathbb{Z}$
- $>$ - - - - - - - $>$ - - - - -
- $\geq$ - - - - - - - - $\geq$ - - - - -
- $=$ - - - - $\{(x,y) \mid x=y\} \subseteq \mathbb{Z}\times\mathbb{Z}$

  $= \{(x,x) \mid x\in\mathbb{Z}\}$

- $\neq$ - - - - $\mathbb{Z}\times\mathbb{Z} - \{(x,x) \mid x\in\mathbb{Z}\}$
- $\mid$ is the set $\{(x,y) \mid x \mid y\} \subseteq \mathbb{Z}\times\mathbb{Z}$

- for any $m \in \mathbb{Z}^+$, $\equiv_m$ is the set $\{(x,y) : m \mid (x-y)\} \subseteq \mathbb{Z} \times \mathbb{Z}$

<u>Ex</u> let $\mathcal{U}$ be a set. $\in$ is a relation from $\mathcal{U}$ to $\mathbb{P}(\mathcal{U})$

$\in$ is the set $\{(x, A) \mid x \in A\}$

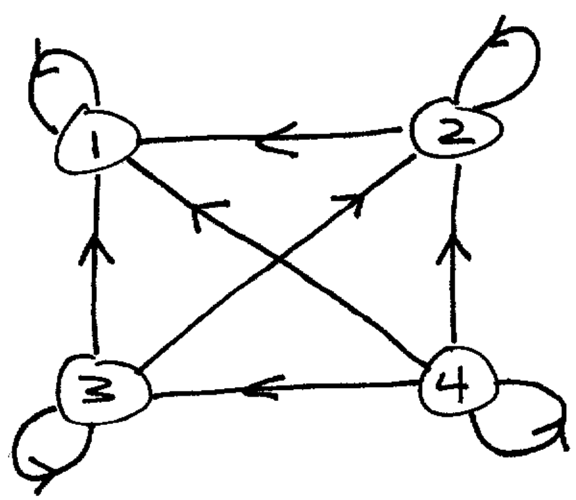
<u>Relations on a finite set</u>

<u>Ex.</u> let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1), (2,1), (2,2), (3,3), (3,2), (3,1),$
$(4,4), (4,3), (4,2), (4,1)\}$

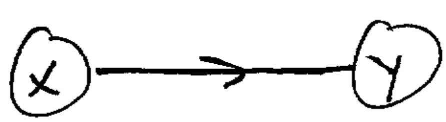
Thus $4 R 1$ and $3 R 2$, but $3 \not R 4$.

<u>Directed graph (digraph) representation</u>



This is the $\geq$ Relation on $A = \{1, 2, 3, 4\}$, i.e. . . . . .

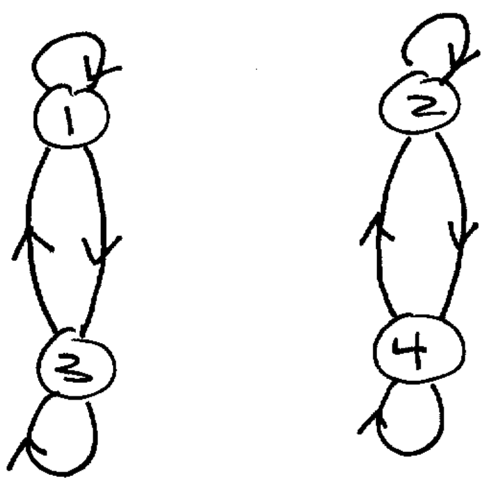
$$x R y \text{ iff } x \geq y$$



Ex. again $A = \{1, 2, 3, 4\}$

$S = \{(1,1), (1,3), (3,1), (3,3), (2,2),$
$(2,4), (4,2), (4,4)\}$

digraph representation :



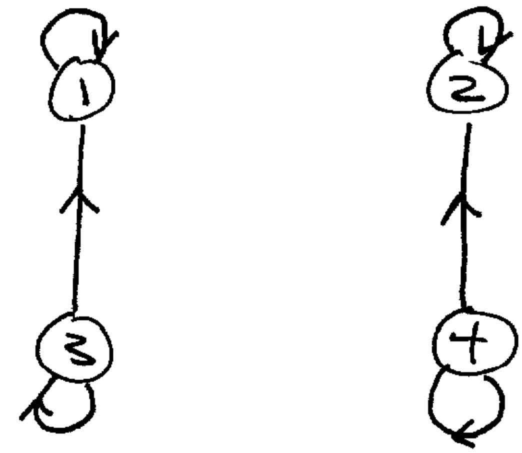
This is the same parity relation

i.e. $x \, S \, y$ iff $x \equiv_2 y$:

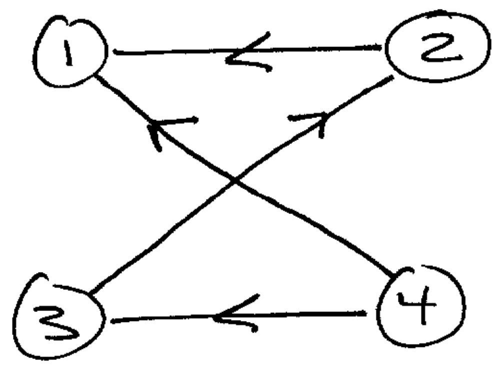
$$S = (\equiv_2)$$

we can perform set operations on relations.

$R \cap S$:



$R - S$:



Exercise

Draw digraph reps. of

$A \times A$ $\quad$ $R \cup S$

$(A \times A) - S$ $\quad$ $R \oplus S$

$(A \times A) - R$ $\quad$ $S - R$

(11.1) hw Problems: 2, 4, 6, 8, 10

(11.2) Properties of relations

Defn

we call a relation $R \subseteq A \times A$

- reflexive iff $\forall x \in A$:

$$x R x$$

- symmetric iff $\forall x, y \in A$:

$$x R y \Rightarrow y R x$$

- transitive iff $\forall x, y, z \in A$

$$x R y \wedge y R z \Rightarrow x R z$$

- antisymmetric iff $\forall x, y \in A$

$$x R y \wedge y R x \Rightarrow x = y$$

Relation	Ref.	Symm.	tran.	antisymm
$<$	no	no	yes	yes
$\leq$	yes	no	yes	yes
$>$	no	no	yes	yes
$\geq$	yes	no	yes	yes
$=$	yes	yes	yes	yes
$\neq$	no	yes	no	no
$\mid$	yes	no	yes	yes
$\equiv_m$	yes	yes	yes	no

Exercise:

Prove all 24 entries above.

Problems (11.2) # 2, 4, 6