cse 16 8-4-21

another variation:

- strong induction with multiple base cases. To show $\forall n \geq n_0 : P(n)$

I. $P(n_0), P(n_0+1), \ldots, P(n_1)$ are true.

IId. $\forall n > n_1 : (P(n_0) \wedge \cdots \wedge P(n-1)) \Rightarrow P(n)$

Ex.

$P(n)$

$\forall n \geq 8 : \boxed{\exists x, y \geq 0 \text{ such that } n = 3x + 5y}$

Note: some of $P(1), P(2), \ldots, P(7)$ are ~~all~~ false.

Proof.

I. $P(8)$ is true since $8 = 3\cdot 1 + 5\cdot 1$

$P(9)$ " " " $9 = 3\cdot 3 + 5\cdot 0$

$P(10)$ " " " $10 = 3\cdot 0 + 5\cdot 2$

II d. $\forall n > 10 : (P(8) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$

Let $n > 10$ be arbitrary. Assume for all k in the range $8 \leq k < n$ that there exist $x_k, y_k \geq 0$ s.t.

$$k = 3x_k + 5y_k .$$

we must show that

$$n = 3x + 5y$$

for some $x, y \geq 0$.

Since $n > 10 \Rightarrow n \geq 11 \Rightarrow n-3 \geq 8$,

The ind. hyp. provides $x' \geq 0$, $y' \geq 0$ such that

$$n-3 = 3x' + 5y'$$

Let $x = x'+1$ and $y = y'$. Then

$$\begin{aligned} n &= (n-3)+3 \\ &= 3x' + 5y' + 3 \quad \left\{ \text{by ind. hyp.} \right. \\ &= 3(x'+1) + 5y' \\ &= 3x + 5y. \end{aligned}$$

Result follows for all $n \geq 8$ by 2nd P.M.I. ∎

When should we use strong induction? It depends on which domino(s) topple the n^{th} domino.

If the domino to be toppled requires anything other than the immediately preceding domino, use strong induction.

Theorem (Binomial Thm.)

Let $x, y \in \mathbb{R}$. Then

$$\forall n \geq 0 : \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

$P(n)$

we use weak ind. form II b, and Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \begin{array}{c}\text{for}\\ 0<k<n\end{array}$$

Proof:

I. $P(0)$ says $(x+y)^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k$,

i.e. $(x+y)^0 = \binom{0}{0} \cdot x^0 y^0$, i.e. $1 = 1$ ✓.

II b. $\forall n>0 : P(n-1) \Rightarrow P(n)$.

Let $n>0$. Assume

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k$$

we must show:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

we have

$$(x+y)^n = (x+y)^{n-1} \cdot (x+y)$$

$$= \left(\sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k \right) \cdot (x+y) \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind.} \\ \text{hyp.} \end{array} \right.$$

$$= x \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k + y \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^k$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k} y^{k+1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-\cancel{1}-(k-\cancel{1})} y^{(k-\cancel{1})+\cancel{1}}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-k} y^k$$

$$= \binom{n-1}{0} x^n y^0 + \sum_{k=1}^{n-1} \binom{n-1}{k} x^{n-k} y^k$$

$$+ \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{n-k} y^k + \binom{n-1}{n-1} x^0 y^n$$

$$= \binom{n-1}{0} x^n y^0 + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] x^{n-k} y^k + \binom{n-1}{n-1} x^0 y^n$$

$$= \binom{n}{0} x^n y^0 + \sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k + \binom{n}{n} x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

Result follows for all $n \geq 0$ by 1^{st} P.M.I. ∎

Recursion

many sequences can be defined efficiently by recursion:

(1) define a finite set of base objects

(2) specify rule(s) for constructing new objects from existing ones.

By applying (1) and (2) repeatedly, we construct a seq. of objects.

Ex. Fibonacci Sequence

$$① \begin{cases} F_1 = 1 \\ F_2 = 1 \end{cases}$$

$$② \{ F_n = F_{n-1} + F_{n-2} \quad (\text{for } n \geq 3)$$

so

$$\begin{array}{ccccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & \cdots \\ \| & \| & \| & \| & \| & \| & \| & \| & \\ (1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & \ldots) \end{array}$$

<u>Ex.</u> $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} F_{2k-1} = F_{2n}} \leftarrow P(n)$

$n=1: \quad F_1 = F_2$ ✓

$n=2: \quad F_1 + F_3 = F_4$ ✓

$n=3: \quad F_1 + F_3 + F_5 = F_6$ ✓

$n=4: \quad F_1 + F_3 + F_5 + F_7 = F_8$ ✓

Proof.

I. $P(1)$ says $1=1$, which is true.

IIb. $\forall n > 1 : P(n-1) \Rightarrow P(n)$.

Let $n > 1$. Assume, for this n, that

$$\sum_{k=1}^{n-1} F_{2k-1} = F_{2n-2} .$$

we must show

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n} .$$

we have

$$\sum_{k=1}^{n} F_{2k-1} = \left(\sum_{k=1}^{n-1} F_{2k-1}\right) + F_{2n-1}$$

$$= F_{2n-2} + F_{2n-1} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind.} \\ \text{hyp.} \end{array} \right.$$

$$= F_{2n} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{Fibonacci Rec.} \end{array} \right.$$

Result follows by 1st P.M.I.

<u>Ex.</u>

$$\forall n \geq 1 : \boxed{\sum_{k=1}^{n} F_k \cdot F_{k+1} = \begin{cases} F_{n+1}^{2} & n \text{ odd} \\ F_n \cdot F_{n+2} & n \text{ even} \end{cases}}$$

$P(n)$

$n = 1$: $F_1 F_2 = F_1^{2}$ ✓

$n = 2$: $F_1 F_2 + F_2 F_3 = F_2 F_4$ ✓

$\vdots$

Proof.

I. $P(1)$ says $F_1 F_2 = F_1^2$, i.e. $1=1$. ✓

II b. $\forall n>1 : P(n-1) \Rightarrow P(n)$.

let $n>1$. Assume

$$\sum_{k=1}^{n-1} F_k \cdot F_{k+1} = \begin{cases} F_n^2 & n \text{ even} \\ F_{n-1}F_{n+1} & n \text{ odd} \end{cases}$$

we must show:

$$\sum_{k=1}^{n} F_k F_{k+1} = \begin{cases} F_{n+1}^2 & n \text{ odd} \\ F_n F_{n+2} & n \text{ even} \end{cases}$$

so

$$\sum_{k=1}^{n} F_k F_{k+1} = \left(\sum_{k=1}^{n-1} F_k F_{k+1} \right) + F_n F_{n+1}$$

$$= \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases} + F_n F_{n+1}$$

$$= \begin{cases} F_n^2 + F_n F_{n+1} & (n \text{ even}) \\ F_{n-1} F_{n+1} + F_n F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_n (F_n + F_{n+1}) & (n \text{ even}) \\ (F_{n-1} + F_n) F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_{n+1}^2 & (n \text{ odd}) \\ F_n F_{n+2} & (n \text{ even}) \end{cases}$$

Result follows by 1^{st} P.M.I.

$\blacksquare$