cse 16 8-12-21

Review Problems

11.1 #2

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(x, y) \in A \times A : x \mid y\}$

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (2,2), (3,3), (3,6), (4,4), (5,5), (6,6)\}$

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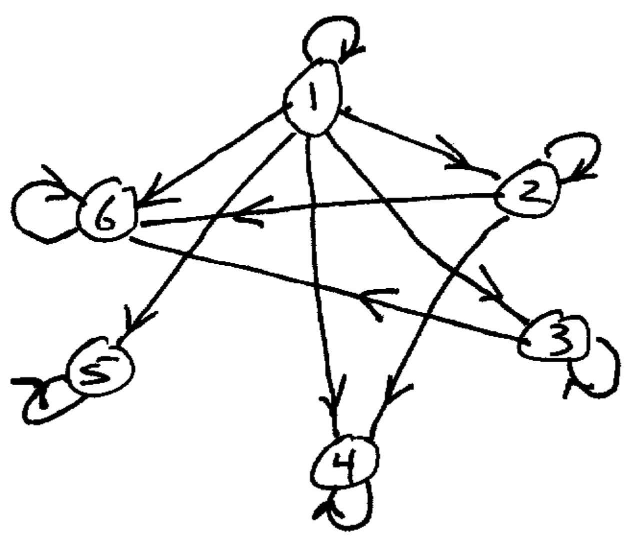

ref. yes

sym. no

trans. yes

antisym. yes

<u>note</u>: $|$ <u>is</u> antisym. on $\mathbb{Z}^+$:

$$x|y \wedge y|x \Rightarrow x=y$$

$$y=k_1x \wedge x=k_2y \Rightarrow x=(k_1k_2)x$$

$$\Rightarrow 1=k_1k_2$$

$$k_1=k_2=1 \qquad k_1=k_2=-1$$

<u>But</u>: $|$ <u>is not</u> antisymm. on $\mathbb{Z}$

since in this case could have $k_1=k_2=-1$, so $y=-x$ not $y=x$.

Additional Prob. in hw7

(1) # relations R on A where $|A|=n$.

$R \subseteq A \times A \qquad |A \times A| = n \cdot n = n^2$

answer $= 2^{n^2}$

matrix rep $\quad n \times n$ matrix with 0, 1

$$M = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \end{array}$$

bit strings of len n^2 $= 2^{n^2}$.

(2) # $R \subseteq A \times A$ that are reflexive

All diag. elements in M are 1

have $n^2 - n$ free bits ~~#~~, so

$\therefore$ ans. $= 2^{n^2 - n}$

(3) # $R \subseteq A \times A$ that are symm.

$$\# \text{ free bits} = \frac{n(n+1)}{2}$$

$$\begin{array}{ll} 1 & * \\ 2 & * \; * \\ 3 & * \; * \; * \\ \vdots & \vdots \\ n & * \; * \; * \; \cdots \; * \end{array}$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\text{ans.} = 2^{\frac{n(n+1)}{2}}$$

(4) # $R \subseteq A \times A$ both ref. & symm.

$$\# \text{ free bits} = \frac{n(n-1)}{2}$$

$$\begin{array}{ll} 1 & \emptyset \\ 2 & * \\ 3 & * \; * \\ \vdots & \\ n & * \; * \; \cdots \; * \end{array} \qquad \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-1 \end{array}$$

$$1+2+\cdots(n-1) = \frac{n(n-1)}{2}$$

$$\text{ans.} = 2^{\frac{n(n-1)}{2}}$$

11.1 #8

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \phi$

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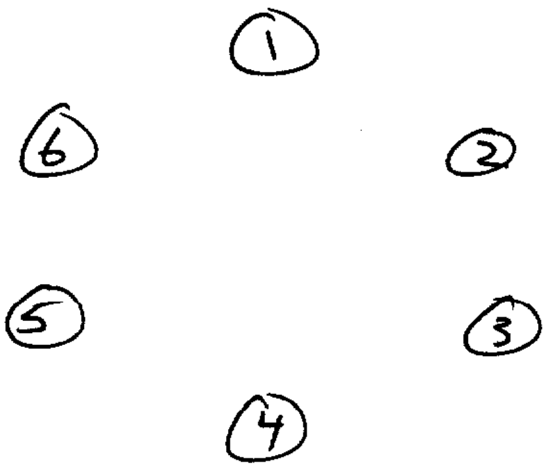

Properties

ref. : $\forall x : xRx$ no

symm : $\forall x, y : xRy \Rightarrow yRx$ yes

trans : $\forall x, y, z : xRy \wedge yRz \Rightarrow xRz$ yes

anti-symm : $\forall x, y : xRy \wedge yRx \Rightarrow x = y$ yes

Ch. 10 #42

F_n is even iff $3 \mid n$.

check:

n	1	2	3	4	5	6	7	8	9	...
F_n	1	1	2	3	5	8	13	21	34	...

proof. (strong IId with 2 base cases).

I $n=1$: $F_1 = 1$ odd and $3 \nmid 1$

$n=2$: $F_2 = 1$ odd and $3 \nmid 2$

II. $\forall n > 2$: $P(1) \wedge P(2) \wedge \ldots \wedge P(n-1) \Rightarrow P(n)$

Let $n > 2$ be arbitrary. $\therefore n \geq 3$.

we have 3 cases: $n = 3k, 3k+1, 3k+2$

for some $k \in \mathbb{Z}$.

<u>case: $n = 3K$</u>

Then $3 \nmid 3K-1 = n-1$ and $3 \mid 3K-2 = n-2$.

by ind. hyp. F_{n-1}, F_{n-2} are odd. $\therefore$ $F_n = F_{n-1} + F_{n-2}$ is even

<u>case: $n = 3K+1$</u>

Then $3 \mid (n-1) = 3K$ and $3 \nmid (n-2) = 3K-1$

$\therefore$ by ind. hyp. F_{n-1} even & F_{n-2} odd.

$\therefore$ $F_n = F_{n-1} + F_{n-2}$ is odd.

<u>case: $n = 3K+2$</u>

Then $3 \nmid (n-1) = 3K+1$ & $3 \mid (n-2) = 3K$

$\therefore$ by ind. hyp. F_{n-1} odd & F_{n-2} even

$\therefore$ $F_n = F_{n-1} + F_{n-2}$ is odd

[redacted]

ch. 10 #28

$$\forall n \geq 1 : \sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$$

I. $n=1$: $F_2 = F_3 - 1$, i.e. $1 = 2 - 1$ ✓

IIa $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$. Assume: $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$.

show: $\sum_{k=1}^{n+1} F_{2k} = F_{2n+3} - 1$. Then

$$\sum_{k=1}^{n+1} F_{2k} = \left(\sum_{k=1}^{n} F_{2k}\right) + F_{2(n+1)}$$

$$= \left(F_{2n+1} - 1\right) + F_{2n+2} \quad \left\{ \text{by the ind. hyp.} \right.$$

$$= \left(F_{2n+1} + F_{2n+2}\right) - 1$$

$$= F_{2n+3} - 1 .$$

Result follows ....

11.3 #2

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6),$
$(2,3), (3,2), (4,5), (5,4), (4,6), (6,4),$
$(5,6), (6,5)\}$

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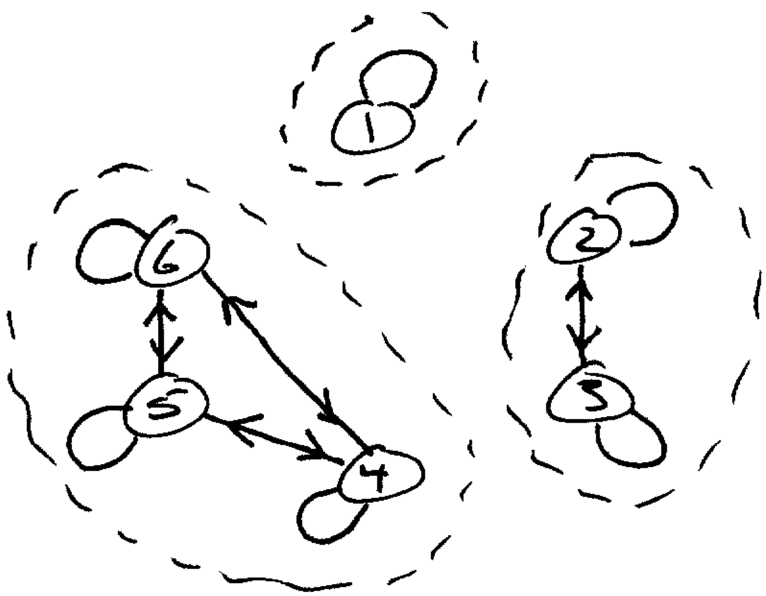

eq. classes:

$[1] = \{1\}$
$[2] = [3] = \{2, 3\}$
$[4] = [5] = [6] = \{4, 5, 6\}$.

11.3 #4

$A = \{a, b, c, d, e\}$

$R \subseteq A \times A$ eq. rel.

aRd, bRc, eRa, eRe

find # of eq. classes.

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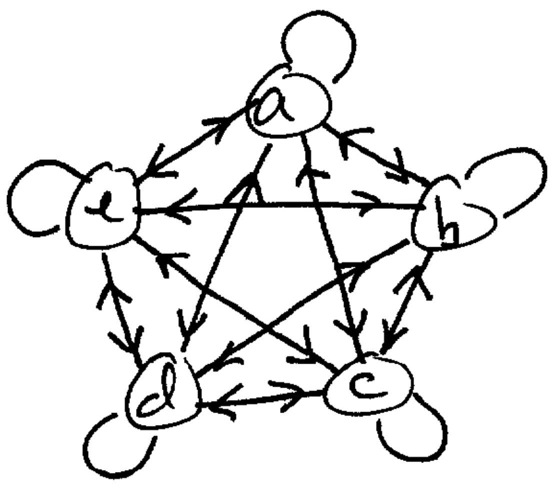


$R = A \times A$

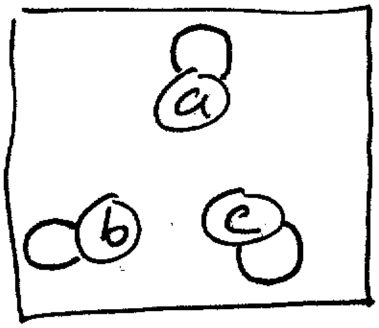
(# eq. classes) = 1

11.3 #6)

describe all 5 eq. rel. on $A=\{a,b,c\}$

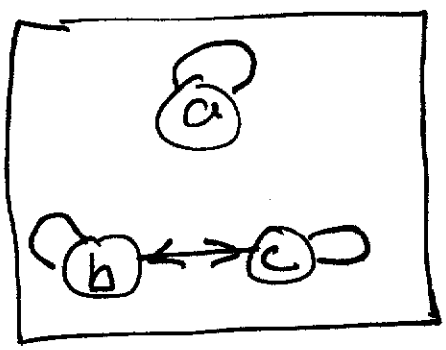
soln:

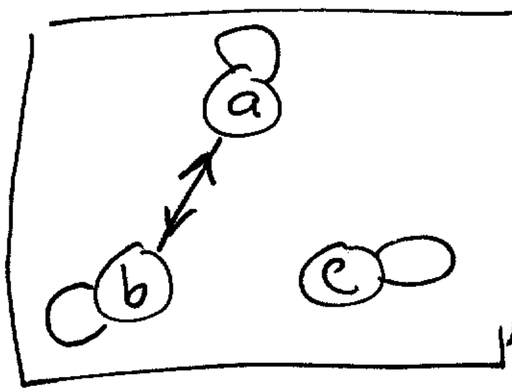
3 eq. classes

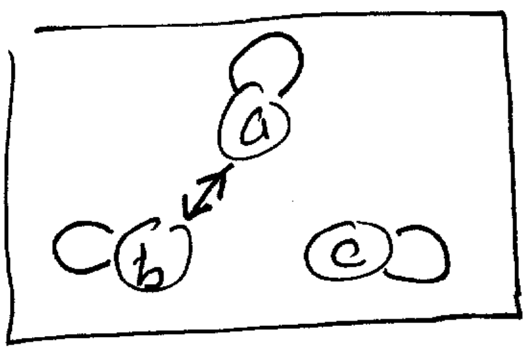


$R = (=)$ $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

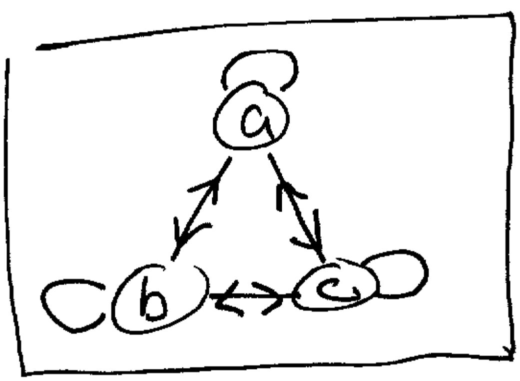
2 eq. classes







1 eq class



$R = (A \times A)$

$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

11.4 #6

$P = \{\{0\}, \{-1, 1\}, \{-2, 2\}, \{-3, 3\}, \ldots\}$

a partition on $\mathbb{Z}$.

describe $R \subseteq \mathbb{Z} \times \mathbb{Z}$ with elts. of P as eq. classes.

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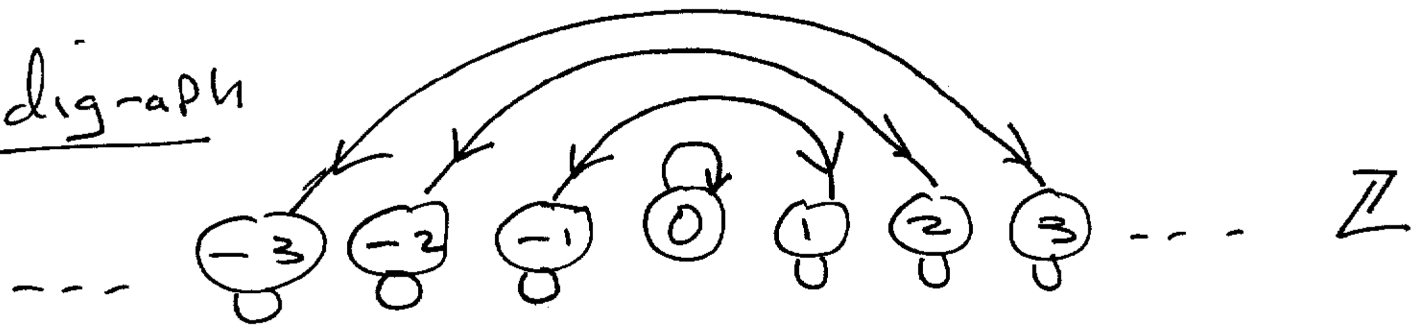

$R = \{(x, y) \mid y = \pm x\}$

$(x-y)(x+y) = 0$

$x^2 - y^2 = 0$

$x^2 = y^2$

$|x| = |y|$

hw & additional Prob

$R = \{ ((a,b),(c,d)) \mid a,b,c,d \in \mathbb{Z},\ b \neq 0,\ d \neq 0$
and $ad = bc \}$

show R is ~~eq.~~ rel on $A = \{(a,b) \mid b \neq 0\}$.

ref : $(a,b) R (a,b)$ since $ab = ab$ ✓

symm. $(a,b) R (c,d) \Rightarrow ad = bc$
$\Rightarrow bc = ad$
$\Rightarrow cb = da$
$\Rightarrow (c,d) R (a,b)$ ✓

trans $(a,b) R (c,d) \wedge (c,d) R (e,f) \Rightarrow$
$\therefore\ ad = bc \wedge cf = de$
want : $af = be$
have $adf = bcf \wedge bcf = bde$

$\therefore \quad adf = bde$

$\therefore \quad af = be \quad (\text{since } d \neq 0)$

$\therefore \quad (a,b)\, R\, (e,f)$. ∎