cse 16 8-10-21

Ex. show $\equiv_m$ is transitive, i.e.

$$x \equiv_m y \land y \equiv_m z \Rightarrow x \equiv_m z$$

Proof.

$$\left.\begin{array}{l} x \equiv_m y \Rightarrow m \mid (x-y) \\ y \equiv_m z \Rightarrow m \mid (y-z) \end{array}\right\} \Rightarrow m \mid [(x-\cancel{y})+(\cancel{y}-z)]$$

$$\Rightarrow m \mid (x-z) \Rightarrow x \equiv_m z .$$

Ex. show $|$ is not symmetric

Proof:

we must show: $\forall x, y: x \mid y \Rightarrow y \mid x$

is a false statement.

i.e. show

$$\sim \forall x, y : x \mid y \Rightarrow y \mid x$$

i.e. show

$$\exists x, y : x \mid y \wedge y \nmid x$$

Let $x = 2$, $y = 4$. Then $2 \mid 4$ but $4 \nmid 2$.

Ex. $<$ is antisymmetric.

must show $\forall x, y$:

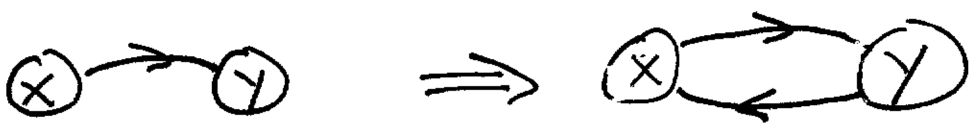
$$(x < y) \wedge (y < x) \Rightarrow (x = y)$$

The hypothesis ~~for~~ all $x, y \in \mathbb{Z}$. Thus the implication is (vacously) true.

Digraph interpretation

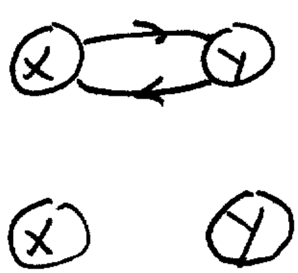
- **reflexive** : $\forall x : xRx$

- **symmetric** $\forall x,y : xRy \Rightarrow yRx$

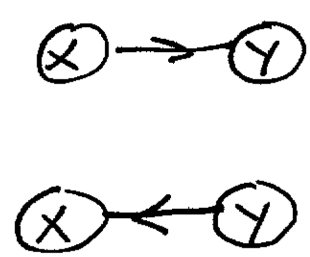


equivalently:

Possible

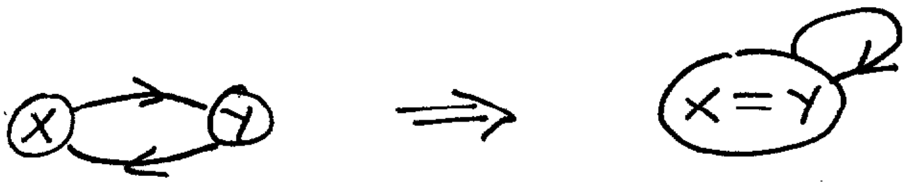


not possible



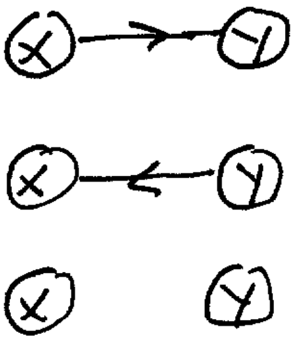
- <u>transitive</u>: $\forall x, y, z: xRy \wedge yRz \Rightarrow xRz$

- <u>antisymmetric</u>: $\forall x, y: xRy \wedge yRx \Rightarrow x=y$

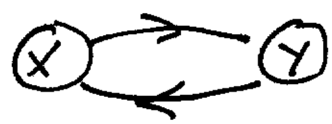


equivalently: given $x \neq y$

<u>possible</u>



<u>not possible</u>

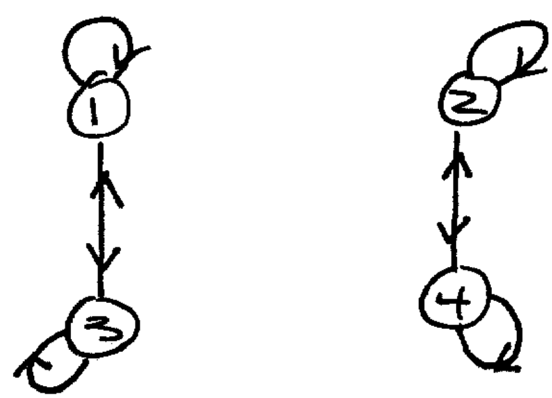


Matrix representation

Ex. $A = \{1, 2, 3, 4\}$ $\quad R = (\equiv_2)$

$R = \{(1,1), (2,2), (3,3), (4,4), (1,3),$
$(3,1), (2,4), (4,2)\}$

digraph



matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

11.3 Equivalence Relations

Defn

A relation $R \subseteq A \times A$ on a set A is called a <u>equivalence relation</u> iff it is:

(1) reflexive
(2) symmetric
(3) transitive

Ex. the table in last sec. says

$$= \quad \{(x,x) \mid x \in \mathbb{Z}\}$$

and

$$\equiv_m \quad \{(x,y) \mid x, y \in \mathbb{Z} \text{ and } m \mid (x-y)\}$$

are equiv. relations.

Lets complete this for $\equiv_m$:

- $\equiv_m$ is reflexive

$m|0 \Rightarrow m|(x-x) \Rightarrow x \equiv_m x$

- $\equiv_m$ is symmetric

$m|a \Rightarrow m|(-a)$ since if $a = km$, then $(-a) = (-k)m$. Hence

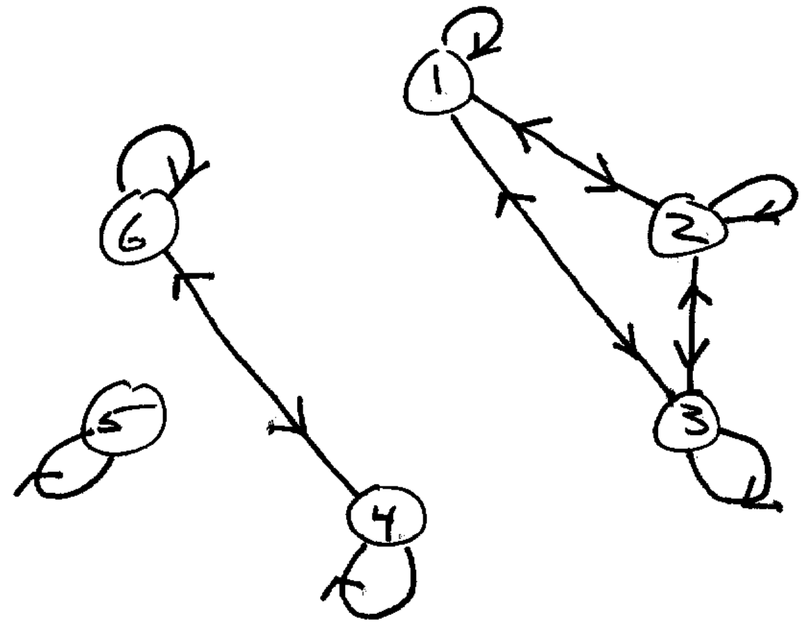
$$\begin{aligned} x \equiv_m y &\Rightarrow m|(x-y) && \text{(defn)} \\ &\Rightarrow m|(y-x) && \text{(above)} \\ &\Rightarrow y \equiv_m x && \text{(defn.)} \end{aligned}$$

- $\equiv_m$ is transitive (shown previously)

<u>Ex.</u> let $A = \{1, 2, 3, 4, 5, 6\}$

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,6), (5,5), (6,4), (6,6)\}$$

<u>digraph</u>



<u>check</u>

ref. ✓

symm. ✓

trans. ✓

<u>matrix</u>

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Defn

Let $R \subseteq A \times A$ be an equivalence relation on A, and let $a \in A$. The equivalence class containing a, denoted $[a]$ is the subset

$$[a] = \{x \in A \mid xRa\} \subseteq A.$$

i.e. $[a]$ is the set of all $x \in A$ that are equiv. to a (under R).

other notation: $[a]_R$

Ex. Previous on $A = \{1, 2, 3, 4, 5, 6\}$

$[1] = [2] = [3] = \{1, 2, 3\}$

$[4] = [6] = \{4, 6\}$

$[5] = \{5\}$

<u>lemma</u>

let R be an eq. rel. on A.

Then for any $a, b \in A$:

$$a R b \text{ iff } [a]=[b]$$

$\Rightarrow$ ✓

$\Leftarrow$ ✓

<u>Proof.</u>

($\Rightarrow$) suppose aRb. It's suff. to show $x \in [a] \Leftrightarrow x \in [b]$, for then $[a]=[b]$.

$$\begin{aligned} x \in [a] &\Rightarrow xRa && \text{(defn)} \\ &\Rightarrow xRa \wedge aRb && \text{(hyp.)} \\ &\Rightarrow xRb && \text{(trans.)} \\ &\Rightarrow x \in [b] && \text{(def.)} \end{aligned}$$

Also

$$x \in [b] \Rightarrow xRb \quad \text{(defn)}$$
$$\Rightarrow xRb \wedge aRb \quad \text{(hyp.)}$$
$$\Rightarrow xRb \wedge bRa \quad \text{(symm.)}$$
$$\Rightarrow xRa \quad \text{(trans.)}$$

we've shown $x \in [a] \Leftrightarrow x \in [b]$, hence $[a] = [b]$.

($\Leftarrow$) suppose $[a] = [b]$. Then

$$aRa \quad \text{(ref.)}$$
$$\therefore a \in [a] \quad \text{(def.)}$$
$$\therefore a \in [b] \quad \text{(hyp.)}$$
$$\therefore aRb \quad \text{(defn.)}$$

<u>Rmk:</u> we used all 3 properties, ref., symm., and trans.

Given $m \in \mathbb{Z}^+$, what are the eq. classes of $\equiv_m$?

By last lemma:

$$
\begin{aligned}
x \in [a] \quad & \text{iff} \quad x \equiv_m a \\
& \text{iff} \quad m \mid (x-a) \\
& \text{iff} \quad x - a = km \quad (\text{some } k \in \mathbb{Z}) \\
& \text{iff} \quad x = a + km \quad (\text{''} \quad)
\end{aligned}
$$

$\therefore \; [a] = \{ a + km \mid k \in \mathbb{Z} \}$