CSE 16 7-8-21

Notation:

If $|S|=n$, the number of k-combinations of S (k-element subsets of S) is denoted

$$C(n,k) \quad \text{or} \quad \binom{n}{k}$$

read "n-choose-k"

Ex. $S=\{1,2,3\}$ $n=3$, $0 \le k \le 3$

		$C(n,k)$
$k=3$:	$\{1,2,3\}$	1
$k=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	3
$k=1$:	$\{1\}, \{2\}, \{3\}$	3
$k=0$:	ϕ	1

Theorem

If $0 \le k \le n$, then

$$\binom{n}{k} = C(n,k) = \frac{n!}{k!\,(n-k)!}$$

Proof.

we compute $P(n,k)$ again, but in a different way.

To construct a k-permutation of an n-element set: $|S| = n$.

- choose a k-element subset of S this can be done in $C(n,k)$ ways.
- choose an ordered arrangement of these k elements. This can be done in $k!$ ways.

The number of ways of making these choices in succession is

$$C(n,k) \cdot k!$$

By the mult. principle

$$P(n,k) = C(n,k) \cdot k!$$

$$\therefore \quad \frac{n!}{(n-k)!} = C(n,k) \cdot k!$$

$$\therefore \quad C(n,k) = \frac{n!}{k!\,(n-k)!} \qquad \blacksquare$$

__Ex.__ how many bit strings of length 10 contain exactly 6 ones?

__note:__ each such bit string encodes a 6-element subset of $\{1,2,3,\ldots,10\}$

$$\frac{1}{1}\ \frac{0}{2}\ \frac{1}{3}\ \frac{0}{4}\ \frac{1}{5}\ \frac{1}{6}\ \frac{0}{7}\ \frac{1}{8}\ \frac{1}{9}\ \frac{0}{10}$$

corresponds to the subset

$$\{1, 3, 5, 6, 8, 9\}$$

i.e. the set of index positions in the string occupied by 1's.

Thus

$$\begin{pmatrix}\#\text{ bit strings of length 10 cont.} \\ \text{exactly 6 ones}\end{pmatrix}$$

$$= \begin{pmatrix}\#\text{ 6-element subsets of} \\ \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\end{pmatrix}$$

$$= \binom{10}{6} = \frac{10!}{6!\,4!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = 10 \cdot 21$$

$$= \boxed{210}$$

note: a bit string of len. 10 cont. exactly 6 1's, must contain exactly 4 0's.

$\therefore$ (# bit str. of len. 10 cont. 4 0's)

= (# bit str. of len. 10 cont. 6 1's)

= $\boxed{210}$.

Also note: choosing a 6-subset of a 10-set is equiv. to choosing a 4-subset of the 10-set, i.e. its complement. Thus

$$\binom{10}{6} = \binom{10}{4}$$

$\binom{10}{6}$: choose which 6 positions to be occupied by 1's

$\binom{10}{4}$: choose which 4 positions are to be occupied by 0's.

indeed

$$\binom{10}{6} = \frac{10!}{6!\,4!} = 210$$

and

$$\binom{10}{4} = \frac{10!}{4!\,6!} = 210$$

Theorem

If $0 \le k \le n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic Proof

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \binom{n}{n-k}$$

Combinatorial Proof:

The LHS counts the # of bit strings of length n containing exactly k ones.

The RHS counts the number of bit strings of length n containing exactly $(n-k)$ zeros.

These two sets of strings are the same! Thus LHS = RHS, i.e.

$$\binom{n}{k} = \binom{n}{n-k}.$$

note: $\binom{n}{k}$ are sometimes called Binomial Coefficients.

(3.6) Binomial Theorem

using identity

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad (0 \le k \le n)$$

we compute:

$$\begin{array}{ll}
\underline{n} & \\
0 & \binom{0}{0} = 1 \\
1 & \binom{1}{0} = 1, \ \binom{1}{1} = 1 \\
2 & \binom{2}{0} = 1, \ \binom{2}{1} = 2, \ \binom{2}{2} = 1 \\
3 & \binom{3}{0} = 1, \ \binom{3}{1} = 3, \ \binom{3}{2} = 3, \ \binom{3}{3} = 1 \\
4 & \binom{4}{0} = 1, \ \binom{4}{1} = 4, \ \binom{4}{2} = 6, \ \binom{4}{3} = 4, \ \binom{4}{4} = 1 \\
\vdots & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{array}$$

Pascal's Δ:

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
$\vdots$	

conditions that Pascal's Δ:

(1) 1's on sides of Δ

(2) interior entries are sum of neighboring entries on row above

<u>Theorem</u> (Pascal's identity)

For $1 \leq k \leq n$

(*) $$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$

Equivently, for $0 < k < n$

(**) $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

<u>Algebraic proof of (*)</u>

$$\text{RHS} = \binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{k \cdot n!}{k \cdot (k-1)!\,(n-k+1)!} + \frac{(n-k+1)\,n!}{k!\,(n-k)!\,(n-k+1)}$$

$$= \frac{k \cdot n!}{k!\,(n-k+1)!} + \frac{(n-k+1) \cdot n!}{k!\,(n-k+1)!}$$

$$= \frac{k \cdot n! + (n-k+1) \cdot n!}{k!\,(n-k+1)!}$$

$$= \frac{(\cancel{k} + n - \cancel{k} + 1) \cdot n!}{k!\,((n+1)-k)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!}$$

$$= \binom{n+1}{k} = \text{LHS}. \qquad \blacksquare$$

Combinatorial proof of $(**)$:

Recall:

$$(**) \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (0 < k < n)$$

Let $S = \{$ bit strings of len. n containing exactly k 1's $\}$

Let $A = \{x \in S \mid x$ ends in a 1 $\}$

$B = \{x \in S \mid x$ ends in a 0 $\}$

clearly $S = A \cup B$ and $A \cap B = \emptyset$.

(also $A \neq \emptyset$ since $k > 0$, $B \neq \emptyset$ since $k < n$.)

By the addition principle

$$|S| = |A| + |B|$$

- Every $x \in A$ can be obtained by choosing a bit string of len. $(n-1)$ containing $(k-1)$ 1's, and appending a 1.

$$x = \underbrace{\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot}_{\substack{\text{len. } n-1 \\ \text{cont. } (k-1) \text{ 1's}}} 1$$

There are $\binom{n-1}{k-1}$ such strings.

- Every $x \in B$ can be obtained by choosing a bit str. of len. $(n-1)$ cont. k 1's, then append 0

$$x = \underbrace{\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot}_{\substack{\text{len. } (n-1) \\ \text{cont. } k \text{ 1's}}} 0$$

There ~~are~~ are $\binom{n-1}{k}$ such strings.

Thus

$$\binom{n}{k} = |S| = |A| + |B|$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}.$$

<u>Exercise</u>

- write an algebraic proof of **.
- write a combinatorial proof of *.

Pascal's Δ appears in another context

n	
0	$(x+y)^0 = 1$
1	$(x+y)^1 = 1\cdot x + 1\cdot y$
2	$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$
3	$(x+y)^3 = 1\cdot x^3 + 3\cdot x^2y + 3\cdot xy^2 + 1\cdot y^3$
4	$(x+y)^4 = 1\cdot x^4 + 4\cdot x^3y + 6\cdot x^2y^2 + 4\cdot xy^3 + 1\cdot y^4$

when $(x+y)^n$ is fully expanded and all like terms are combined, each term is of the form

$$\binom{n}{k} x^{n-k} y^k \quad (0 \leq k \leq n)$$

<u>Binomial Theorem</u>

for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Algebraic proof: later, when we've covered induction.

Combinatorial Proof

Number each factor in $(x+y)^n$

$$(x+y)^n = \underbrace{\overset{1}{(x+y)}\overset{2}{(x+y)}\overset{3}{(x+y)}\cdots\cdots\overset{n}{(x+y)}}_{n \text{ factors}}$$

Continue .....