CSE 16 7-7-21

<u>Ex.</u> Bit Strings

A <u>bit string</u> is a string the alphabet $\{0, 1\}$.

how many bit strings of length n.

$$\begin{array}{lccccc} & 1 & 2 & 3 & \cdots & n \\ & \underline{1} & \underline{0} & \underline{1} & \cdots & \underline{0} \\ \text{\# choices:} & \underbrace{2 \cdot 2 \cdot 2 \quad \cdots \quad \cdot 2}_{n} & & & & \end{array} = \boxed{2^n}$$

<u>Ex.</u> The # of strings of length n from an alphabet of size a is

$$\begin{array}{lccccc} & 1 & 2 & 3 & \cdots & n \\ \text{\#choice} & \overline{a} \cdot & \overline{a} \cdot & \overline{a} \cdot & & \cdot \overline{a} \end{array} = \boxed{a^n}$$

Ex how many such strings have no repeated letter?

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ \_ & \_ & \_ & & \_ \end{array}$$

choices: $a \quad (a-1) \quad (a-2) \quad \cdots \quad (a-n+1)$

$$\#\text{strings} = \boxed{a\cdot(a-1)(a-2)\cdots(a-n+1)}$$

$$= \boxed{\frac{a!}{(a-n)!}}$$

(3.3) Addition & Subtraction Principles

Ex how many strings of length 4 from alphabet $\{a, b, c, d, e, f\}$ are there satisfying :

- no repeated letters
- must contain an 'a'

There are 4 types of such strings

I. $\underline{a}\ \underset{5}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{3}{\underline{\quad}}$ $\qquad 5 \cdot 4 \cdot 3 = 60$

II. $\underset{5}{\underline{\quad}}\ \underline{a}\ \underset{4}{\underline{\quad}}\ \underset{3}{\underline{\quad}}$ $\qquad 60$

III $\underset{5}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underline{a}\ \underset{3}{\underline{\quad}}$ $\qquad 60$

IV $\underset{5}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \underline{a}$ $\qquad 60$

strings $= 60 + 60 + 60 + 60$

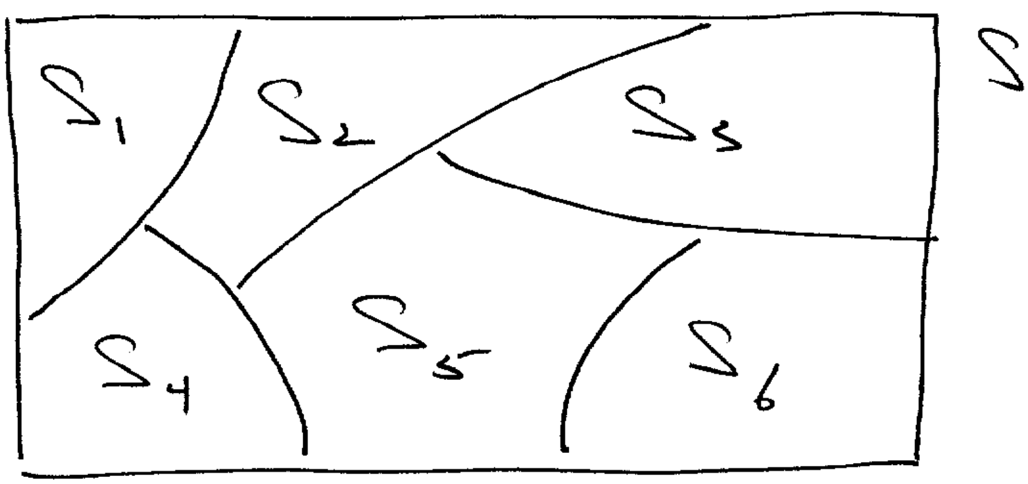
$= \boxed{240}$

Defn

A Partition of a set S is a collection of (non-empty) subsets $S_1, S_2, \ldots, S_n$ of S satisfying

(1) $S_i \cap S_j = \phi$ for $1 \leq i < j \leq n$

(2) $\bigcup_{i=1}^{n} S_i = S$



Addition Principle

If $S_1, S_2, \ldots, S_n$ is a Partition of a finite set S, then

$$|S| = |S_1| + |S_2| + \cdots + |S_n|.$$

note: formula is correct even if some S_i are empty.

Ex. how many 4-digit numbers satisfy

- even. (last digit is even $\{2,4,6,8\}$)
- no 0's (i.e. alphabet $= \{1,2,3,4,5,6,7,8,9\}$)
- exactly one 5.
- exactly one 4, appearing left of 5.

we have 3 sets of numbers

		# choices
S_1:	$\underset{1}{4}\ \underset{1}{5}\ \underset{7}{\cdot}\ \underset{3}{\cdot}$	$7 \cdot 3 = 21$
S_2:	$\underset{1}{4}\ \underset{7}{\_}\ \underset{1}{5}\ \underset{3}{\_}$	21
S_3:	$\underset{7}{\_}\ \underset{1}{4}\ \underset{1}{5}\ \underset{3}{\_}$	21

$$\# \text{strings} = 21 + 21 + 21 = \boxed{63}$$

observe for any subset $A \subseteq S$ with $\emptyset \neq A \subsetneq S$, we have a partition of S into two subsets

$$A, \ \bar{A} = S - A$$

$$\therefore \quad |S| = |A| + |\overline{A}|$$

$$= |A| + |S - A|$$

$$\therefore \quad |S - A| = |S| - |A|$$

<u>note</u>: Also holds if $A = \phi$, or $A = S$

<u>Subtraction Principle</u>

if S is finite and $A \subseteq S$,

then

$$|S - A| = |S| - |A|$$

Ex. how many strings of length 6 from $\{a,b,c,d,e\}$ contain at least 1 'd' ?

$S = \{\text{strings len}=6 \text{ over } \{a,b,c,d,e\}\}$

$A = \{\text{strings in } S \text{ with } \underline{\text{no}} \text{ 'd's}\}$

$= \{\text{strings of len}=6 \text{ over } \{a,b,c,e\}\}$

want: $|S-A|$

$$|S-A| = |S| - |A|$$

$$= 5^6 - 4^6$$

$$= 15{,}625 - 4{,}096$$

$$= \boxed{11{,}529}$$

(3.4) Permutations

Defn

A <u>permutation</u> of a finite set S is an ordered arrangement of all of its elements

<u>note</u>: same as a list or string with no repetition.

<u>Ex</u> $S = \{1, 2, 3\}$ $\#$

Permutations of S:

$\{123, 132, 213, 231, 312, 321\}$ $\boxed{6}$

<u>Ex</u> $S = \{1, 2\}$

Permutations: $12, 21$ $\boxed{2}$

Theorem

If $|S| = n$, then the number of permutations of S is

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

follows from mult. principle.

Defn

A <u>k-permutation</u> of a finite set S is an ordered arrangement of k of its elements. $(0 \leq k \leq |S|)$

Ex. $S = \{1, 2, 3\}$ $\quad n = 3$

		#
$k=3$:	123, 132, 213, 231, 312, 321	6
$k=2$:	12, 21, 13, 31, 23, 32	6
$k=1$:	1, 2, 3	3
$k=0$:	$\emptyset$	1

Notation: the number of k-permutations of an n-element set is denoted

$$P(n, k)$$

observe: $P(n, 0) = 1$

$$P(n, n) = n!$$

also: if $k > n$, then $P(n, k) = 0$

By the mult. principle, we have

<u>Theorem</u>:

$$P(n,k) = \begin{cases} n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!} & 0 \le k \le n \\ 0 & k > n \end{cases}$$

<u>Ex.</u> List all $P(4,3) = 4 \cdot 3 \cdot 2 = 24$ 3-permutations of $\{1, 2, 3, 4\}$.

123	124	134	234
132	142	143	243
213	214	314	324
231	241	341	342
312	412	413	423
321	421	431	432

Ex (3.4 #7)

find # of 9-digit numbers satisfying

- no zeros $\{1, 2, 3, \dots, 9\}$
- no repetition
- all odd digits to left of all even digits.

Solution.

odd digits $= \{1, 3, 5, 7, 9\}$
even digits $= \{2, 4, 6, 8\}$

- form str. of len 5 from odd digits
- form " " " 4 " even digits

$$\# \text{numbers} = 5! \cdot 4!$$
$$= (120) \cdot (24) = \boxed{2{,}880}$$

(3.5) Combinations (subsets)

<u>Defn</u>

A <u>k-combination</u> of a finite set S is a k-element subset of S, i.e. an unordered arrangement of k of its elements