Ex. Let $x \in \mathbb{R}$, $x \neq 1$. show

$$\forall n \geq 1 : \boxed{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}} \leftarrow P(n)$$

Proof.

I. $P(1)$ says $\sum_{k=0}^{0} x^k = \frac{x^1 - 1}{x - 1}$, i.e.

$$x^0 = \frac{x-1}{x-1}$$

$$\therefore \quad 1 = 1,$$

which is true.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

Let $n \geq 1$ be chosen arbitrarily.

Assume

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} . \qquad \text{(ind. hyp.)}$$

we must show

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} . \qquad \text{(ind. conclusion.)}$$

so

$$\sum_{k=0}^{n} x^k = \left(\sum_{k=0}^{n-1} x^k \right) + x^n$$

$$= \frac{x^n - 1}{x - 1} + x^n \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{x^n - 1 + x^n(x - 1)}{x - 1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x - 1}.$$

By the P.M.I. we have

$$\forall n \geq 1 : \sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}. \blacksquare$$

<u>Remarks</u>

- always state: "let $n \geq 1$" in II.
- always state ind. hyp. $P(n)$ explicitely.
- always state ind. concl. $P(n+1)$ explicitely.
- always state the point(s) at which the ind. hyp. is used.

Ex. $\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}} \Leftarrow P(n)$

Proof

I. $P(1)$ says $\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$, i.e.

$1 = \frac{1 \cdot 2}{2}$, i.e. $1 = 1$, which is true.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

let $n \geq 1$. Assume for this n that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

we must show

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}.$$

so

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2} .$$

Result follows for all $n \geq 1$ by P.M.I.

<u>Exercise</u> (hw)

$$\forall n \geq 1 : \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Rmk: There's a clever purely algebraic proof of $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof. (not induction)

let

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n.$$

$$\therefore \quad S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$\therefore \quad S + S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$\therefore \quad 2S = n(n+1)$$

$$\therefore \quad S = \frac{n(n+1)}{2}. \quad \blacksquare$$

Ex.

$$\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2}$$

Proof.

I. $P(1)$ says $\displaystyle\sum_{k=1}^{1} k^3 = \left(\frac{1\cdot(1+1)}{2}\right)^2$,

i.e. $1^3 = \left(\frac{1\cdot\cancel{2}}{\cancel{2}}\right)^2$, i.e. $1^3 = 1^2$,

i.e. $1 = 1$, which is true.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$ be arbitrary. Assume, for this n that

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

we must show

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

so

$$\sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n} k^3\right) + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

By the P.M.I., we have

$$\forall n \geq 1 : \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad \blacksquare$$

<u>Rmks</u>

- Induction isn't always about proving summation formula.
- sometimes $P(n)$ is false for a finite # of initial terms.

$P(1), P(2), \ldots, P(n_0-1)$ are false.

our goal is to prove:

$$\forall n \geq n_0 : P(n)$$

I. show $P(n_0)$ is true

II. $\forall n \geq n_0 : P(n) \Rightarrow P(n+1)$.

<u>Ex</u>. $\forall n \geq 4$: $\boxed{5n+8 < n^2+4n+1}$ ← $P(n)$

(check: $P(1), P(2), P(3)$ are false.)

<u>Proof</u>.

<u>I</u>. $P(4)$ says $5 \cdot 4+8 < 4^2+4 \cdot 4+1$,
i.e. $28 < 33$, which is true.

<u>II</u>. $\forall n \geq 4$: $P(n) \Rightarrow P(n+1)$.

Let $n \geq 4$ be arbitrary. Assume

$$5n+8 < n^2+4n+1 \qquad \text{(ind. hyp.)}$$

we must show

$$5(n+1)+8 < (n+1)^2+4(n+1)+1 \qquad \text{(ind. concl.)}$$

Observe

$$\begin{aligned}
5(n+1)+8 &= 5n+5+8 \\
&= (5n+8)+5 \\
&< (n^2+4n+1)+5 && \left\{\begin{array}{l}\text{by the}\\ \text{ind. hyp.}\end{array}\right. \\
&= n^2+4n+6 \\
&< n^2+4n+6+2n && \left\{\begin{array}{l}\text{since } n\geq 4\\ \Rightarrow 2n \geq 8 > 0\end{array}\right. \\
&= n^2+6n+6 \\
&\vdots \\
&= (n+1)^2+4(n+1)+1 && \left\{\begin{array}{l}\text{by some}\\ \text{algebra}\\ \text{(exercise)}\end{array}\right. .
\end{aligned}$$

Result follows for all $n \geq 4$

by P.M.I. $\blacksquare$

Ex. (ch. 10 #17)

Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of subsets of $\mathcal{U}$. Show

$$\forall n \geq 1 \quad \boxed{\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}} \leftarrow P(n)$$

Rmk. This is DeMorgan's law extended to n sets.

Proof:

I. $P(1)$ says

$$\overline{\bigcap_{k=1}^{1} A_k} = \bigcup_{k=1}^{1} \overline{A_k}$$

i.e. $\overline{A_1} = \overline{A_1}$,

which is true.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

Let $n \geq 1$. Assume

$$\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k} \qquad \text{(ind. hyp.)}$$

we must show that

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \bigcup_{k=1}^{n+1} \overline{A_k} \qquad \text{(ind. concl.)}$$

so

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \overline{\left(\bigcap_{k=1}^{n} A_k\right) \cap A_{n+1}}$$

$$= \left(\overline{\bigcap_{k=1}^{n} A_k}\right) \cup \overline{A_{n+1}} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{2-set} \\ \text{version} \\ \text{of DeMorg.} \\ \text{law} \end{array} \right.$$

$$= \left(\bigcup_{k=1}^{n} \overline{A_k} \right) \cup \overline{A_{n+1}} \quad \text{by the ind. hyp.}$$

$$= \bigcup_{k=1}^{n+1} \overline{A_k} .$$

Result follows for all $n \geq 1$ by P.M.I. $\blacksquare$

<u>Exercise</u> (ch. 10 #18)

$$\forall n \geq 1 : \overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k} .$$

Exercise

Let $P_1, P_2, P_3, \ldots$ be an infinite sequence of statements.

notation:

$$\bigvee_{k=1}^{n} P_k = P_1 \vee P_2 \vee P_3 \vee \cdots \vee P_n$$

$$\bigwedge_{k=1}^{n} P_k = P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n$$

Prove

(1) $\sim\left(\bigvee_{k=1}^{n} P_k\right) = \bigwedge_{k=1}^{n} (\sim P_k)$

(2) $\sim\left(\bigwedge_{k=1}^{n} P_k\right) = \bigvee_{k=1}^{n} (\sim P_k)$