Goal: show there exist irrational numbers x, y such that x^y is rational, (constructively!).

Lemma

$\log_2(9)$ is irrational.

Proof:

Assume $\log_2(9) \in \mathbb{Q}$. This means

$$\log_2(9) = \frac{a}{b}$$

for some $a, b \in \mathbb{Z}$ $(b \neq 0)$. Then

$$2^{\log_2(9)} = 2^{\frac{a}{b}}$$

$$\therefore \quad 9 = 2^{\frac{a}{b}}$$

$$\therefore \quad 9^b = \left(2^{\frac{a}{b}}\right)^b = 2^{\frac{a}{\cancel{b}}\cdot \cancel{b}} = 2^a$$

$$\therefore \quad 9^b = 2^a$$

But LHS is odd, while RHS is even. This $\times$ shows our assumption was false, i.e

$$\log_2(9) \in \mathbb{R} - \mathbb{Q}.$$

$\blacksquare$

<u>Proof of thm:</u>

Let $x = \sqrt{2}$ and $y = \log_2(9)$.

we've shown both x, y are irrational. Then

$$
\begin{aligned}
x^{y} &= \sqrt{2}^{\log_2(9)} = \sqrt{2}^{\log_2(3^2)} \\
&= \sqrt{2}^{2\log_2(3)} \\
&= \left(\sqrt{2}^{2}\right)^{\log_2(3)} \\
&= 2^{\log_2(3)} \\
&= 3 \in \mathbb{Q}.
\end{aligned}
$$

$\blacksquare$

Exercise

Prove same thing using:

- $x = \sqrt{3}$, $y = \log_3(25)$
- $x = \sqrt[3]{2}$, $y = \log_2(27)$

Theorem

Let $a, b \in \mathbb{Z}^+$. Then there exist $n, m \in \mathbb{Z}$ for which

$$\gcd(a,b) = an + bm$$

Proof.

Let

$$S = \{ax + by \mid x, y \in \mathbb{Z}\}$$

Note S contains both positive and negative integers, and zero. However $S \cap \mathbb{Z}^+$ contains only positives, and $S \cap \mathbb{Z}^+ \neq \phi$.

Let d be the least positive element in S. (exists by the the W.O.P. of $\mathbb{Z}^+$.)

we will show that $d = \gcd(a, b)$.

By defn of S, there will exist $n, m \in \mathbb{Z}$ such that

$$\gcd(a,b) = d = an + bm.$$

we must show

(i) $d \mid a$ and $d \mid b$

and (ii) if $e \mid a$ and $e \mid b$, then $e \le d$

<u>Proof of (i)</u>

to prove $d \mid a$, use div. algorithm

$$a = qd + r \qquad (0 \le r < d)$$

$$\therefore \quad r = a - qd$$
$$= a - q(an + bm)$$
$$= a(1 - qn) + b(-qm)$$

we see r has the form $r = ax + by$, and hence $r \in S$. But

$$0 \le r < d$$

since d is the Least Positive element of S, we must have $r=0$. Therefore

$$a = q \cdot d$$

and hence $d \mid a$.

exercise: show $d \mid b$.

Proof of (ii)

suppose $e \in \mathbb{Z}^+$ with $e \mid a$ and $e \mid b$.

$\therefore \; e \mid (an+bm)$

$\therefore \; e \mid d \quad \therefore \; e \leq d$.

Therefore $d = \gcd(a,b)$, as required. ∎

Chapter 10 : Mathematical Induction

Ex. observe

$1 = 1^2$

$1 + 3 = 2^2$

$1 + 3 + 5 = 3^2$

$1 + 3 + 5 + 7 = 4^2$

$\vdots$

Guess: for any $n \in \mathbb{Z}^+$:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

other notation :-

$$\sum_{k=1}^{n} (2k-1) = n^2$$

Any particular $n \in \mathbb{Z}^+$ gives an easily verified fact of arithmetic. How to prove all of them?

$$\forall n \in \mathbb{Z}^+ : \sum_{k=1}^{n} (2k-1) = n^2 .$$

Let $P(n)$ be a propositional fcn on universe $\mathbb{Z}^+$. Suppose we wish to prove the statement

$$(*) \qquad \forall n \in \mathbb{Z}^+ : P(n)$$

A proof of $(*)$ by mathematical induction has 2 steps.

I. Base

Prove that $P(1)$ is true.

II. Induction

Prove: $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

i.e. let $n \geq 1$ be an arbitrary pos. int. Assume

$P(n)$ $\left\{ \begin{array}{l} \text{called the} \\ \underline{\text{induction hypothesis}} \end{array} \right.$

is true. show as a consequence that

$P(n+1)$ $\left\{ \begin{array}{l} \text{we call this the} \\ \underline{\text{induction conclusion}} \end{array} \right.$

is true. Hence $P(n) \Rightarrow P(n+1)$.

since $n \geq 1$ was arbitrary:

$$\forall n \geq 1 : P(n) \Rightarrow P(n+1).$$

Once I. and II. are complete, conclude that (*)

$$\forall n \geq 1 : P(n)$$

is also true.

<u>Remark</u>

Assuming $P(n)$ true in induction step is not circular reasoning.

Intuition;

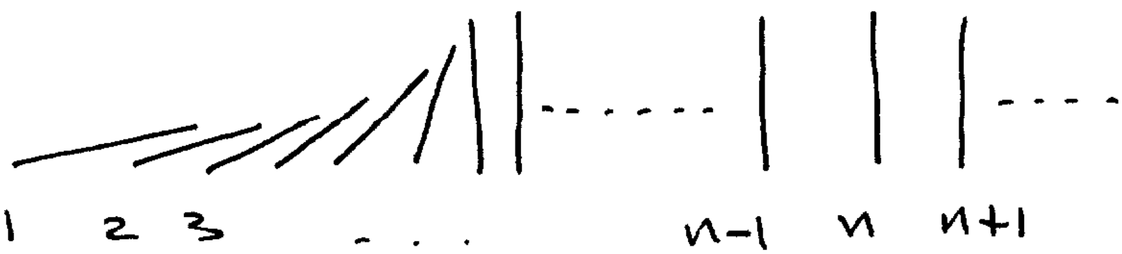
Imagine a proof proceeding as follows:

$P(1)$	(by I.)
$P(1) \Rightarrow P(2)$	(by II with $n=1$)
$P(2)$	(by $\Rightarrow$)
$P(2) \Rightarrow P(3)$	(by II with $n=2$)
$P(3)$	(by $\Rightarrow$).

$\vdots$

The problem with this sequence is that its infinite, so its not a proof.

Domino Analogy



$P(n) =$ "the n^{th} domino falls".

I. $P(1)$ is true: "1^{st} domino falls".

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$:

"if any particular domino falls, then the next one also falls."

<u>Conclusion</u>: $\forall n \geq 1 : P(n)$

"all dominos fall".

The validity of Induction follows from

<u>Theorem</u> (Principle of Mathematical Induction) (PMI)

For any propositional functions $P(n)$ on the universe $\mathbb{Z}^+$, the following is true:

$$\left[P(1) \wedge \forall n\left(P(n) \Rightarrow P(n+1)\right)\right] \Rightarrow \forall n\, P(n)$$

<u>Ex.</u> $\forall n \geq 1 : \boxed{\displaystyle\sum_{k=1}^{n} (2k-1) = n^2}$

$P(n)$

Proof

I. $P(1)$ is the statement

$$\sum_{k=1}^{1} (2k-1) = 1^2$$

i.e. $2 \cdot 1 - 1 = 1^2$

i.e. $1 = 1$

which is true.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$ be arbitrary. Assume that

$$\sum_{k=1}^{n} (2k-1) = n^2 .$$

we must show that

$$\sum_{k=1}^{n+1} (2k-1) = (n+1)^2 .$$

we have

$$\sum_{k=1}^{n+1} (2k-1) = \left(\sum_{k=1}^{n} (2k-1)\right) + \left(2(n+1)-1\right)$$

$$= n^2 + \left(2(n+1)-1\right) \quad \left\{ \text{by the ind. hyp.} \right.$$

$$= n^2 + 2n + 1$$

$$= (n+1)^2 , \quad \text{as required}$$

By the PMI, we conclude

$$\forall n \geq 1 : \sum_{k=1}^{n} (2k-1) = n^2$$

$\blacksquare$