CSE 16 7-15-21

Ex. find the smallest n such that in any group of n people, at least 10 were born in the same month.

By P.P. (a) if n objects (people) are placed in 12 boxes (months), then some box (month) contains at least $\lceil \frac{n}{12} \rceil$ objects (people).

we seek smallest n s.t.

$$\left\lceil \frac{n}{12} \right\rceil = 10$$

$$\therefore \quad 9 < \frac{n}{12} \leq 10$$

$$\therefore \quad 9 \cdot 12 < n \leq 120$$

The smallest such n is

$$n = 9 \cdot 12 + 1 = 108 + 1 = \boxed{109}$$

Ex.

Suppose 8 numbers are chosen from $\{1, 2, 3, \ldots, 14\}$. Show that at least 2 of the 8 numbers must sum to 15.

Brute-force: check all $\binom{14}{8} = 3003$ 8-subsets of $\{1, 2, \ldots, 14\}$ to see that they all have this property.

Instead:

Form 7 labeled boxes:

(1, 14) (2, 13) (3, 12) (4, 11) (5, 10) (6, 9) (7, 8)

each of the 8 numbers is placed in the box with that number in its label. By P.P. (a), some box contains at least $\lceil \frac{8}{7} \rceil = 2$ numbers. those 2 numbers sum to 15. $\blacksquare$

Remarks

- If $n > k$ then $\lceil \frac{n}{k} \rceil \geq \frac{n}{k} > 1$, so some box cont. (at least) 2 objects.
- If $n < k$ then $\lfloor \frac{n}{k} \rfloor \leq \frac{n}{k} < 1$, so some box cont. no objects

Generalization of prev. example:

What is the smallest n such that any n-element subset of $\{1, 2, 3, \dots, 2k\}$ contains 2 numbers that sum to $2k+1$.

__Answer__: $n = k+1$

__Exercise__: Prove this using P.P. (a)

(3.10) Combinatorial Proofs

Ex. (3.6 # 10)

show: $$k\binom{n}{k} = n\binom{n-1}{k-1}$$

see soln. to hw3.

Rmk abstractly, both sides count the number of pairs in

$$\{(x,A) \mid x \in A \subseteq S \text{ and } |A| = k\}$$

where $|S| = n$.

Ex (Generalize 3.10 #3)

show

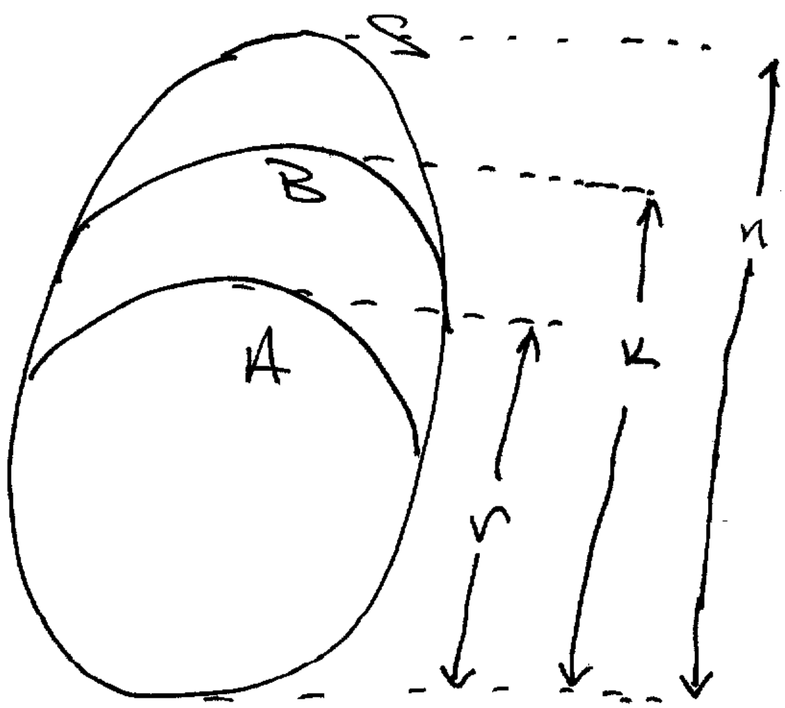
$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$$

where $0 \le r \le k \le n$.

Proof:

let $|S| = n$. we count the set

$$\{(A,B) \mid A \subseteq B \subseteq S,\ |A| = r,\ |B| = k\}$$



I. (1) choose $B \subseteq S$ with $|B| = k$

$\#\text{ways} = \binom{n}{k}$

(2) choose $A \subseteq B$ with $|A| = r$

$\#\text{ways} = \binom{k}{r}$

By mult. Prin. set of $\{(A,B)\}$ has size $\binom{n}{k}\binom{k}{r}$.

II. (1) choose $A \subseteq S$ with $|A| = r$

$\#\text{ways} = \binom{n}{r}$

(2) choose $C \subseteq S - A$ with $|C| = k-r$

$\#\text{ways} = \binom{n-r}{k-r}$

let $B = A \cup C$. $\therefore |B| = r + (k-r) = k$

and $\therefore A \subseteq B$.

By mult. Princ., set $\{(A,B)\}$ has size $\binom{n}{r}\cdot\binom{n-r}{k-r}$ $\blacksquare$

Ex. (3.10 #5)

show $\binom{2n}{2} = 2\binom{n}{2} + n^2$

Let $|S| = 2n$, and let $A \subseteq S$ with $|A| = n$. Then $|\bar{A}| = |S - A| = n$.

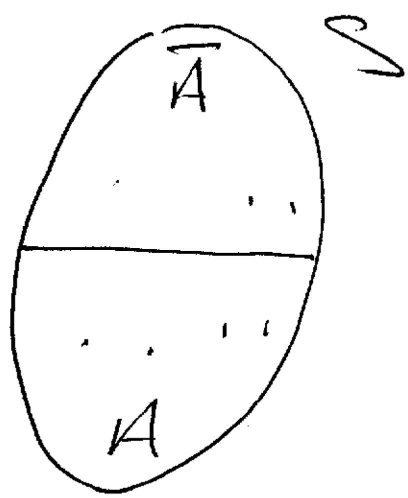
The # of subsets of S of size 2 is obviously $\binom{2n}{2}$, LHS.

To count all $\{x, y\} \subseteq S$ $(x \neq y)$, we partition the set of 2-subsets into 2 classes.

(1) $\{x, y\} \subseteq S$ where x, y belong to the 'same' half. i.e. either $\{x, y\} \subseteq A$ or $\{x, y\} \subseteq \bar{A}$.

The number of such pairs is $\binom{n}{2} + \binom{n}{2} = 2 \cdot \binom{n}{2}$

(2) $\{x, y\} \subseteq A$ with x, y in 'different' halves, i.e. either $x \in A$, $y \in \bar{A}$ or $x \in \bar{A}$, $y \in A$.



ways of choosing $x \in A$ and then $y \in \bar{A}$ is: $n \cdot n = n^2$

By the addition principle, the # of 2-element subsets of S is

$$2\binom{n}{2} + n^2 \qquad \text{(RHS)}.$$

$\blacksquare$

Exercise (3.10 #6)

show

$$\binom{3n}{3} = n^3 + 6n\binom{n}{2} + 3\binom{n}{3}$$

Ex. (P. 109)

show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof:

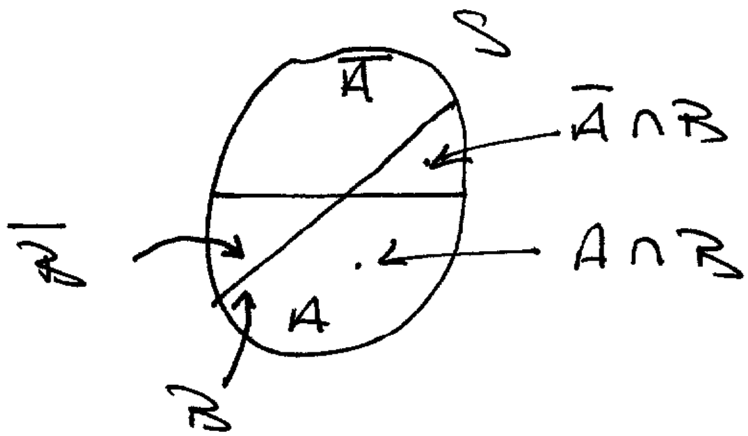
Let $A \subseteq S$ with $|S| = 2n$, $|A| = n$.

Then $|\bar{A}| = n$ also. consider

$$\{B \subseteq S \mid |B| = n\}$$

obviously this set has $\binom{2n}{n}$ subsets.

The task of constructing $B \subseteq S$ with $|B| = n$ partitions into subtasks:



# elements in A	# elements in $\bar{A}$	# ways
0	n	$\binom{n}{0} \cdot \binom{n}{n}$
1	$n-1$	$\binom{n}{1} \cdot \binom{n}{n-1}$
2	$n-2$	$\binom{n}{2} \binom{n}{n-2}$
$\vdots$		
$n-2$	2	$\binom{n}{n-2} \binom{n}{2}$
$n-1$	1	$\binom{n}{n-1} \binom{n}{1}$
n	0	$\binom{n}{n} \cdot \binom{n}{0}$

Thus $\left|\{B \subseteq S : |B| = n\}\right|$ is equal to (by add. Principle)

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} \quad \left\{ \text{since } \binom{n}{n-k} = \binom{n}{k} \right.$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 .$$

Therefore

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

$\blacksquare$

Chapter 4: Direct Proof

Many theorems are of the form

$$P \Rightarrow Q$$

where P, Q are statements.

A *direct proof* of $P \Rightarrow Q$ proceeds by assuming P is true, then try to show Q follows as a consequence.

Next time: more number theory.