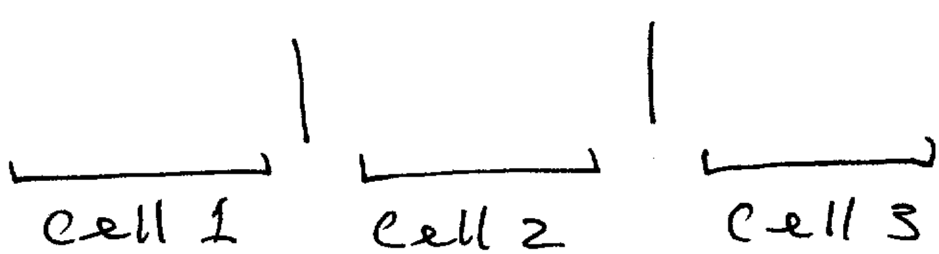
cse 16 7-14-21

<u>multiset</u>	<u>encoding</u>	<u>string</u>
[2, 2, 3, 3]	$\longleftrightarrow$	|**|**
[3, 3, 3, 3]	$\longleftrightarrow$	||****

Each string of length $6 = 2+4 = (3-1)+4$ consists of 2 bars & 4 stars. The 2 bars divide string into 3 cells



cell 1 contains # stars = # 1's
" 2 " " " = # 2's
" 3 " " " = # 3's.

In general if

$$S = \{x_1, x_2, \ldots, x_n\}$$

the k-element multisets based on S are encoded as

$$\underbrace{*\cdots*}_{\substack{\text{cell 1} \\ \uparrow \\ \#\text{ of 1's}}} \; \Big| \; \underbrace{*\cdots*}_{\substack{\text{cell 2} \\ \uparrow \\ \#\text{2's}}} \; \Big| \; \cdots \; \Big| \; \underbrace{*\cdots*}_{\substack{\text{cell } n \\ \uparrow \\ \#\, n\text{'s}}}$$

observe

$$(\# *\text{'s}) = \sum_{i=1}^{n} \text{mult}(x_i) = k$$

$$(\# |\text{'s}) = (\#\text{cells}) - 1 = n - 1$$

Theorem

The number of k-element multisets based on an n-element set equals the number of strings over $\{*, |\}$ of length $k+n-1$ containing k stars (or $n-1$ bars), namely

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}.$$

Prev. example $n = 3$

$k=0$: $\binom{0+3-1}{0} = \binom{2}{0} = 1$

$k=1$: $\binom{1+3-1}{1} = \binom{3}{1} = 3$

$k=2$: $\binom{2+3-1}{2} = \binom{4}{2} = 6$

$k=3: \binom{3+3-1}{3} = \binom{5}{3} = 10$

$k=4: \binom{4+3-1}{4} = \binom{6}{4} = 15$

Ex. (3.19 on P. 99)

have unlimited supply of Red, Blue and Green marbles. How many ways can you select 20 marbles? (consider like colored marbles to be indistinguishable)

Each collection of 20 marbles is a 20-multiset based on $\{R, B, G\}$.

i.e. $n=3$, $k=20$.

$$\binom{20+3-1}{20} = \binom{22}{20} = \boxed{231}$$

<u>note</u> since we're selecting 20 marbles, we could start with 60 marbles! 20 Red, 20 Blue, 20 Green.

<u>Ex.</u>

how many non-negative integer solutions are there to

$$x + y + z = 20$$

<u>note</u>: a solution is a triple $(x, y, z) \in \mathbb{N}^3$ satisfying above eqn.

e.g. $(3, 10, 7)$, $(5, 0, 15)$ <u>are</u> solns and $(\frac{1}{2}, \frac{5}{2}, 17)$ and $(3, 10, 9)$ <u>are not</u> solutions.

Answer : $\boxed{231}$ since this is marble problem again.

<u>soln</u>

$$(3,10,7) \leftrightarrow [\underbrace{R,R,R}_{3}, \underbrace{B,\ldots,B}_{10}, \underbrace{G,\ldots,G}_{7}]$$

$$\leftrightarrow \underbrace{***}_{3} | \underbrace{*\cdots*}_{10} | \underbrace{*\cdots*}_{7}$$

<u>Ex.</u> how many non-negative int. solns. to $x+y+z=20$ satisfy: $x \geq 0, y \geq 0, z \geq 5$.

Let $w = z-5$, then $w \geq 0$ and $z = w+5$. We have

$$x + y + z = 20$$

$$x + y + (z - 5) = 20 - 5$$

$$x + y + w = 15$$

where $x \geq 0$, $y \geq 0$, $w \geq 0$.

$$\#\text{solns} = \binom{15+3-1}{15} = \binom{17}{15} = \boxed{136}$$

Permutations in some elements are indistinguishable (& some are).

also called Anagrams

Ex. Anagrams of LOOK are:

$$\left.\begin{array}{lll} KLOO & LOOK & OKLO \\ LKOO & KOOL & OLKO \\ KOLO & OKOL & OOLK \\ LOKO & OLOK & OOKL \end{array}\right\} \boxed{12}$$

how to count without listing?

Temporarily make the O's distinguishable

$$\{L, O_1, O_2, K\}$$

The # of permutations of is

$$4! = 4 \cdot 3 \cdot 2 = \boxed{24}$$

Divide by # of permutations of $\{O_1, O_2\}$

$$2! = 2 \cdot 1 = \boxed{2}$$

$\therefore$ # anagrams $= \dfrac{24}{2} = \boxed{12}$

i.e. identify:

$$L\,O_1\,O_2\,K \leftrightarrow L\,O_2\,O_1\,K$$

$$K\,O_1\,L\,O_2 \leftrightarrow K\,O_2\,L\,O_1$$

Ex. how many anagrams of

MISSISSIPPI

are there?

$$M\,I_1\,S_1\,S_2\,I_2\,S_3\,S_4\,I_3\,P_1\,P_2\,I_4$$

$\#$ Permutations $= 11!$

$\#$ perm. of $\{I_1, I_2, I_3, I_4\} = 4!$

$\#$ perm. of $\{S_1, S_2, S_3, S_4\} = 4!$

$\#$ " " $\{P_1, P_2\} = 2!$

$$\#\text{anagrams} = \frac{11!}{4!\,4!\,2!} = 34{,}650$$

<u>note</u>: an anagram of

MISSISSIPPI

is a permutation (i.e. an ordering) of the 11-element multiset

$$[I, I, I, I, M, P, P, S, S, S, S]$$

based on $\{I, M, P, S\}$

Theorem

The number of permutations of an n-element multiset based on a k-element set with multiplicities $p_1, p_2, p_3, \ldots, p_k$ $(p_1 + p_2 + \cdots + p_k = n)$ is

$$\frac{n!}{p_1!\, p_2! \cdots p_k!}$$

In mississipi example:

$$\frac{11!}{4!\, 1!\, 2!\, 4!} = 34,650$$

<u>notation</u>!

multinomial coefficient

$$\binom{n}{p_1, p_2, p_3, \ldots, p_k} = \frac{n!}{p_1!\, p_2! \cdots p_k!}$$

where $p_1 + p_2 + p_3 + \cdots + p_k = n$.

These appear in <u>multinomial theorem</u>

(not covered)

<u>Thm</u>

$$\binom{n}{p_1, p_2, \ldots, p_k} = \binom{n}{p_1} \cdot \binom{n-p_1}{p_2} \binom{n-p_1-p_2}{p_3} \cdots \binom{n-p_1-p_2-\cdots-p_{k-1}}{p_k}$$

<u>Exercise</u>: Prove this

- algebraic! easy
- combinatorial! also easy!

(3.9) Pigeonhole Principle

Defn

let $x \in \mathbb{R}$

- $\lfloor x \rfloor$ = greatest int. less than or eq. to x (floor)

- $\lceil x \rceil$ = least int. greater than or eq. to x (ceiling)

Alternately: $\lfloor x \rfloor$ and $\lceil x \rceil$ are the unique ints satisfying

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$

and

- $\lceil x \rceil - 1 < x \leq \lceil x \rceil$

equivalently:

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

Pigeonhole Principle

Suppose n objects are placed in k boxes. Then

(a) at least one box contains $\lceil \frac{n}{k} \rceil$ (or more) objects

(b) at least one box contains $\lfloor \frac{n}{k} \rfloor$ (or fewer) objects.

Proof.

The average # of objects per box is $\frac{n}{k}$. We can't have all boxes containing below the average # objects.

$\therefore$ some box contains $\geq \frac{n}{k}$ objects

Since #obj in a box is an integer,

$\therefore$ some box contains $\geq \left\lceil \frac{n}{k} \right\rceil$ objects

This proves (a).

Similarly, not all boxes have a number of objects above average.

$\therefore$ some box contains $\leq \frac{n}{k}$

$\therefore$ " " " $\leq \left\lfloor \frac{n}{k} \right\rfloor$