CSE 16 7-13-21

Binomial Theorem

Let $n \geq 0$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

comb. Proof:

Number the factors in $(x+y)^n$

$$(x+y)^n = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)} \cdots \underset{n}{(x+y)}$$

when this prod. is fully expanded and before any like terms are combined, each term is a string of length n over alphabet $\{x, y\}$.

for instance

$$(x+y)^3 = \overset{1}{(x+y)}\overset{2}{(x+y)}\overset{3}{(x+y)}$$

$$= xxx + xxy + xyx + xyy$$
$$+ yxx + yxy + yyx + yyy$$

$$= x^3 + x^2y + x^2y + xy^2 + x^2y$$
$$+ xy^2 + xy^2 + y^3$$

any terms with same # of x's and y's can be combined. The coeffient of $x^{n-k}y^k$ $(0 \le k \le n)$ _after_ like terms are combined is the number of ways of choosing which k factors from $\{1, 2, 3, \ldots, n\}$ will contribute a y. This is the # of k-subsets of the n-set $\{1, 2, \ldots, n\}$.

This number is $\binom{n}{k}$. Thus

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \qquad \blacksquare$$

Ex. find coeff. of $x^8 y^5$ in $(x+y)^{13}$

$$\binom{13}{5} = \frac{13!}{5!\,8!} = \frac{13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

$$= \boxed{1,287}$$

Ex. find coeff. of x^5 in $(x-1)^{10}$

$$(-1)^5 \cdot \binom{10}{5} = -\frac{\cancel{10}^{\,3} \cdot \cancel{9}^{\,2} \cdot 8 \cdot 7 \cdot 6}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

$$= -3 \cdot 2 \cdot 7 \cdot 6 = \boxed{-252}$$

Ex let $x = y = 1$ in B.T.

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 1^k$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

Ex. (3.6 #9)

show $$9^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot 10^{n-k}$$

Proof.

let $x = 10$, $y = -1$. Then

$$9^n = (10-1)^n = \sum_{k=0}^{n} \binom{n}{k} 10^{n-k} \cdot (-1)^k .$$

$\blacksquare$

Exercise

In full expansion of

$$\left(x + \frac{1}{x}\right)^{100}$$

a.) determine coeff. of x^{50}

b.) " " " x^{-50}

Exercise

In full expansion of $\left(xy^2 - \frac{1}{z}\right)^{1000}$

a.) determine coeff of $x^{12}y^{13}z^{-975}$

b.) " " " $x^{12}y^{12}z^{-976}$

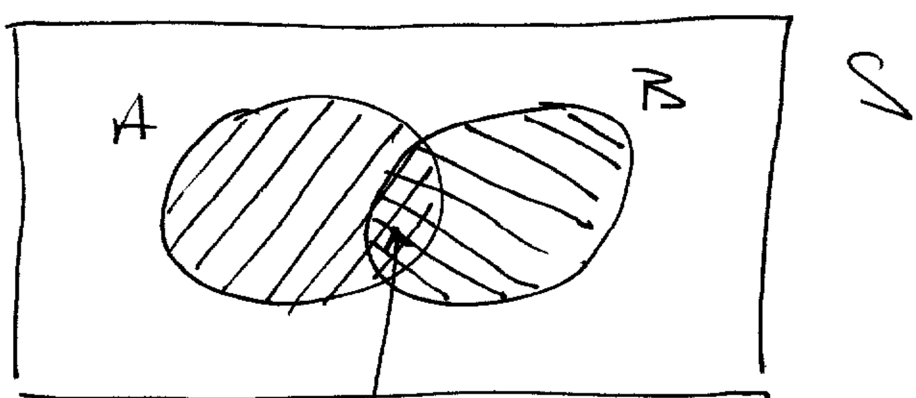
c.) " " " $x^{200}y^{400}z^{-800}$

Answers: (a) 0, (b) 0, (c) $\binom{1000}{800}$

(3.7) Inclusion-Exclusion Principle (PIE)

Recall: $A, B \subseteq S$ finite, then so are $A \cup B$ & $A \cap B$, and

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$A \cap B$

Ex. how many 5-card Poker hands are there?

$\# \text{suits} = 4 \quad \{\underbrace{\heartsuit \ \diamondsuit}_{\text{red}} \ \underbrace{\clubsuit \ \spadesuit}_{\text{black}}\}$

$\# \text{kinds} = 13 \quad \{2, 3, \ldots, 10, J, Q, K, A\}$

$\# \text{cards} = 4 \cdot 13 = 52$

$$\binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960$$

Ex. how many 5-card Poker hands contain at least 1 red ace?

Let

$A = \{$ 5-card P.H. with $A\diamondsuit \}$

$B = \{$ " " " " $A\heartsuit \}$

$A \cap B = \{$ " " " with $A\diamondsuit$ and $A\heartsuit \}$

we seek $|A \cup B|$.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A| = |B| = \binom{51}{4} = 249{,}900$$

$$|A \cap B| = \binom{50}{3} = 19{,}600$$

$\therefore$ $|\{\text{with at least 1 red Ace}\}|$

$$= |A \cup B| = 2 \cdot (249{,}900) - (19{,}600)$$

$$= \boxed{480{,}200}$$

PIE for 3 sets: A, B, C

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

PIE for 4 sets: A, B, C, D

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- \left(|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D| \right)$$
$$+ \left(|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \right)$$
$$- |A \cap B \cap C \cap D|$$

PIE for n sets: $A_1, A_2, \ldots, A_n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

$$- \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

$$\vdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Ex. Suppose $|A| = |B| = |C| = 20$, $|A \cup B \cup C| = 42$, $|A \cap B| = |A \cap C|$, and $B \cap C = \emptyset$. determine $|A \cap B|$.

Let $x = |A \cap B|$. Then

$$42 = 20 + 20 + 20 - x - x - 0 + 0$$

$$\therefore \quad 2x = 60 - 42 = 18$$

$$\therefore \quad \boxed{x = 9.}$$

(3.8) Multisets

set: unordered collection of objects $\{x_1, x_2, \ldots, x_n\}$. no multiple memberships.

list: ordered collection of objects. $(x_1, x_2, \ldots, x_n)$, multiple memberships allowed.

multiset: unordered collection of objects $[x_1, x_2, \ldots, x_n]$. multiple memberships <u>are</u> allowed.

Ex. $A = [1,1,1,2,3,3,4,4,4]$

multiplicity of 1 : 3

" " 2 : 1

" " 3 : 2

" " 4 : 3

$\overline{\quad 9} = |A|$

Defn

The cardinality of a multi-set is denoted $|A|$, and is the number of elements, counting multiplicities

$$|A| = \sum_{\substack{\text{distinct} \\ x \in A}} \text{multiplicity}(x)$$

Defn

The underlying set of a multiset A is set of members of A, without multiplicities.

Ex. multi-sets based on $\{1, 2, 3\}$

Cardinality		#
0 :	$[\ ] = \emptyset$	$\boxed{1}$
1 :	$[1], [2], [3]$	$\boxed{3}$
2 :	$[1,1], [1,2], [1,3], [2,2]$ $[2,3], [3,3]$	$\boxed{6}$
3 :	$[1,1,1], [1,1,2], [1,1,3]$ $[1,2,2], [1,2,3], [1,3,3]$ $[2,2,2], [2,2,3], [2,3,3]$ $[3,3,3]$	$\boxed{10}$

4 : [1,1,1,1], [1,1,1,2], [1,1,1,3]

[1,1,2,2], [1,1,2,3], [1,1,3,3]

[1,2,2,2], [1,2,2,3], [1,2,3,3] $\boxed{15}$

[1,3,3,3], [2,2,2,2], [2,2,2,3]

[2,2,3,3], [2,3,3,3], [3,3,3,3]

<u>Goal</u>: develop a formula for the number of k-element multisets based on an n-element set. ($n \geq 0$ and $k \geq 0$).

we encode such a multiset as a string over alphabet

$$\{*, |\}$$

i.e. a bit string.

special case : $n=3$, $k=4$

multiset	encoding	string
[1, 2, 2, 3]	$\longleftrightarrow$	* | ** | *
[1, 2, 3, 3]	$\longleftrightarrow$	* | * | **
[1, 3, 3, 3]	$\longleftrightarrow$	* | | ***

⋮

continue