

Ex. (tautology)

$$P \vee \sim P$$

✓

P	$\sim P$	$P \vee \sim P$
0	1	1
1	0	1

$$P \Rightarrow P$$

✓

P	$P \Rightarrow P$
0	1
1	1

$$P \Rightarrow (P \vee Q)$$

✓

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

Defn

A contradiction is a (compound) proposition that is false for every assignment of truth values to its Prop. vars.

Ex. (contradiction)

$P \wedge \neg P$
✓

P	$\neg P$	$P \wedge \neg P$
0	1	0
1	0	0

Defn

A statement that is not a contradiction is called Satisfiable.

Ex. P

Defn

A statement that is neither a tautology nor a contradiction is called a contingency

Ex $\rightarrow$

Defn

Two compound statements X, Y are said to be logically equivalent iff the statement

$$X \Leftrightarrow Y$$

is a tautology.

notation : $X = Y$

note: if we write (in algebra)

$$a^2 - b^2 = (a-b)(a+b)$$

we're using = sign in same way!

Ex $P \Rightarrow Q = \neg P \vee Q$ (implication law)

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Same!

Ex. $P \Rightarrow Q = \neg Q \Rightarrow \neg P$ (contrapositive law) 15

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

↔ same ↔

Ex. $P \Rightarrow Q \neq Q \Rightarrow P$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

↔ different ↔

Exercise

Prove all equivalences on p. 52 of B.O.P.

Proof of 2nd DeMorgan

$$\neg(P \vee Q) = \neg P \wedge \neg Q \quad \checkmark$$

P	Q	P ∨ Q	¬(P ∨ Q)	¬P	¬Q	¬P ∧ ¬Q
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

same

Can Prove other identities using
Known identities:

$$\underline{\text{Ex}} \quad (P \vee Q) \Rightarrow R = (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

$$(P \vee Q) \Rightarrow R = \sim(P \vee Q) \vee R \quad (\text{implication law})$$

$$= (\sim P \wedge \sim Q) \vee R \quad (1^{\text{st}} \text{ De Morgan})$$

$$= (\sim P \vee R) \wedge (\sim Q \vee R) \quad \left(\begin{array}{l} \text{comm.} \\ \text{dist.} \end{array} \right)$$

$$= (P \Rightarrow R) \wedge (Q \Rightarrow R) \quad (\text{imp. law twice})$$

other identities

Domination laws:

$$P \vee T = T$$

$$P \wedge F = F$$

Identity laws:

$$P \wedge T = P$$

$$P \vee F = P$$

Idempotent laws:

$$P \vee P = P$$

$$P \wedge P = P$$

Double negation:

$$\neg(\neg P) = P$$

Absorption laws:

$$P \vee (P \wedge Q) = P$$

$$P \wedge (P \vee Q) = P$$

Negation laws:

$$P \vee \neg P = T$$

$$P \wedge \neg P = F$$

Exercise

Prove all of these using truth tables.

Ex. show $(P \wedge Q) \Rightarrow (P \Rightarrow Q)$ is a tautology

$$(P \wedge Q) \Rightarrow (P \Rightarrow Q) = \sim(P \wedge Q) \vee (\sim P \vee Q) \quad (\text{imp. law})$$

$$= (\sim P \vee \sim Q) \vee (\sim P \vee Q) \quad (\text{De Morgan.})$$

$$= (\sim P \vee \sim P) \vee (\sim Q \vee Q) \quad \left(\begin{array}{l} \text{comm.} \\ \text{assoc.} \end{array} \right)$$

$$= \sim P \vee \top \quad (\text{Idem., Negation})$$

$$= \top \quad (\text{Domination})$$

Exercise:

show $\sim(P \Rightarrow Q) \Rightarrow P$ is a tautology using algebra.

Exercise

Determine whether

$$(P \vee \neg Q) \wedge (Q \vee \neg R) \wedge (R \vee \neg P)$$

is satisfiable. (Do it algebraically?)

Set Theory & Logic

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Let A, B, C be sets. Then recall definitions

$$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$$

$$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$$

$$A \oplus B = \{x \in U \mid (x \in A) \oplus (x \in B)\}$$

$$\bar{A} = \{x \in U \mid \sim (x \in A)\} = \{x \in U \mid x \notin A\}$$

$$\emptyset = \{x \in U \mid F\}$$

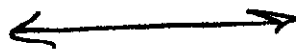
$$U = \{x \in U \mid T\}$$

Analogy:

set theory

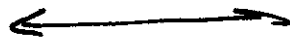
logic

sets



statements

$\cap$



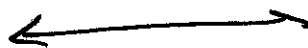
$\wedge$

$\cup$



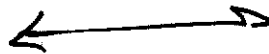
$\vee$

$-$ (complement)



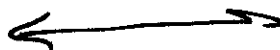
$\sim$ (negation)

$\emptyset$



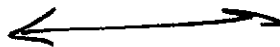
$\top$

U



$\top$

$\oplus$
symm. diff



$\oplus$
xor

so we can prove set identities using known logical identities

Ex De Morgan's laws for sets.

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Proof:

$$\begin{aligned} \overline{A \cup B} &= \{x \in U \mid \sim x \in A \cup B\} \\ &= \{x \in U \mid \sim (x \in A \vee x \in B)\} \\ &= \{x \in U \mid \sim (x \in A) \wedge \sim (x \in B)\} \\ &= \{x \in U \mid x \in \bar{A} \wedge x \in \bar{B}\} \\ &= \bar{A} \cap \bar{B} \end{aligned}$$

Exercise:

Prove the set identities:

De Morgan

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Commutative

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Distributive

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Domination

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

Identity

$$A \cap U = A$$

$$A \cup \emptyset = A$$

Idempotent

$$A \cup A = A$$

$$A \cap A = A$$

Double complement

$$\overline{\overline{A}} = A$$

Absorption

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Complement

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \phi$$