CSE 16 6-30-21

Ex. (tautology)

$P \vee \sim P$ ✓

P	$\sim P$	$P \vee \sim P$
0	1	1
1	0	1

$P \Rightarrow P$ ✓

P	$P \Rightarrow P$
0	1
1	1

$P \Rightarrow (P \vee Q)$ ✓

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

Defn

A contradiction is a (compound) proposition that is false for every assignment of truth values to its prop. vars.

Ex. (contradiction)

$p \wedge \sim p$ ✓

p	$\sim p$	$p \wedge \sim p$
0	1	0
1	0	0

Defn

A statement that is not a contradiction is called satisfiable.

Ex. p

Defn

A statement that is neither a tautology nor a contradiction is called a contingency

Ex p

Defn

Two compound statements X, Y are said to by logically equivalent iff the statement

$$X \Leftrightarrow Y$$

is a tautology.

notation: $X \equiv Y$

<u>note:</u> if we write (in algebra)

$$a^2 - b^2 = (a-b)(a+b)$$

we're using = sign in same way!

<u>Ex</u> $P \Rightarrow Q \;=\; \sim P \vee Q$ (<u>implication law</u>)

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim P \vee Q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

same!

Ex. $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$ (contrapositive law)

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

same

Ex. $P \Rightarrow Q \neq Q \Rightarrow P$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

different

Exercise:

Prove all equivalences on P. 52 of B.O.P.

Proof of 2^{nd} DeMorgan

$\sim(P \vee Q) \equiv \sim P \wedge \sim Q$ ✓

P	Q	$P \vee Q$	$\sim(P \vee Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

same

Can prove other identities using known identities:

Ex $(P \vee Q) \Rightarrow R = (P \Rightarrow R) \wedge (Q \Rightarrow R)$

$$
\begin{aligned}
(P \vee Q) \Rightarrow R &= \sim(P \vee Q) \vee R && \text{(implication law)} \\
&= (\sim P \wedge \sim Q) \vee R && (1^{\text{st}} \text{ Demorgan}) \\
&= (\sim P \vee R) \wedge (\sim Q \vee R) && \left(\begin{matrix}\text{comm. \&} \\ \text{dist.}\end{matrix}\right) \\
&= (P \Rightarrow R) \wedge (Q \Rightarrow R) && \left(\begin{matrix}\text{imp. law} \\ \text{twice}\end{matrix}\right)
\end{aligned}
$$

Other identities

Domination laws:

$$P \vee T = T$$

$$P \wedge F = F$$

Identity laws:

$$P \wedge T = P$$
$$P \vee F = P$$

Idempotent laws:

$$P \vee P = P$$
$$P \wedge P = P$$

Double negation:

$$\sim(\sim P) = P$$

Absorption laws:

$$P \vee (P \wedge Q) = P$$
$$P \wedge (P \vee Q) = P$$

Negation laws:

$$P \vee \sim P = T$$
$$P \wedge \sim P = F$$

Exercise

Prove all of these using truth tables.

Ex. show $(P \wedge Q) \Rightarrow (P \Rightarrow Q)$ is a tautology

$$(P \wedge Q) \Rightarrow (P \Rightarrow Q) = \sim(P \wedge Q) \vee (\sim P \vee Q) \quad \text{(IMP. law)}$$

$$= (\sim P \vee \sim Q) \vee (\sim P \vee Q) \quad \text{(De Morg.)}$$

$$= (\sim P \vee \sim P) \vee (\sim Q \vee Q) \quad \text{(comm. \& assoc.)}$$

$$= \sim P \vee T \quad \text{(Idem., negation)}$$

$$= T \quad \text{(Domination)}$$

Exercise:

show $\sim(P \Rightarrow Q) \Rightarrow P$ is a tautology using algebra.

Exercise

Determine whether

$$(P \vee \sim Q) \wedge (Q \vee \sim R) \wedge (R \vee \sim P)$$

is satisfiable. (Do it algebraically?)

Set Theory & Logic

Let A, B, C be sets. Then recall definitions

$$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$$

$$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$$

$$A \oplus B = \{x \in U \mid (x \in A) \oplus (x \in B)\}$$

$$\bar{A} = \{x \in U \mid \sim (x \in A)\} = \{x \in U \mid x \notin A\}$$

$$\phi = \{x \in U \mid F\}$$

$$U = \{x \in U \mid T\}$$

Analogy:

set theory		logic
sets	$\longleftrightarrow$	statements
$\cap$	$\longleftrightarrow$	$\wedge$
$\cup$	$\longleftrightarrow$	$\vee$
$-$ (complement)	$\longleftrightarrow$	$\sim$ (negation)
$\emptyset$	$\longleftrightarrow$	F
$\mathcal{U}$	$\longleftrightarrow$	T
$\oplus$ symm. diff	$\longleftrightarrow$	$\oplus$ exor

so we can prove set identities using known logical identities

<u>Ex</u> De morgan's laws for sets.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

<u>Proof</u>:

$$\begin{aligned}
\overline{A \cup B} &= \{x \in U \mid \sim x \in A \cup B\} \\
&= \{x \in U \mid \sim (x \in A \vee x \in B)\} \\
&= \{x \in U \mid \sim (x \in A) \wedge \sim (x \in B)\} \\
&= \{x \in U \mid x \in \overline{A} \wedge x \in \overline{B}\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}$$

<u>Exercise</u>:

Prove the set identities :

Demo-gan

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

commutative

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Distributive

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Domination

$$A \cup U = U$$

$$A \cap \phi = \phi$$

Identity

$$A \cap U = A$$

$$A \cup \phi = A$$

Idempotent

$$A \cup A = A$$

$$A \cap A = A$$

Double complement

$$\overline{\overline{A}} = A$$

Absorption

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Complement

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \phi$$