CSE 16 6-24-21

Suppose $J \neq \phi$ and $J \subseteq I$.

Does it follow that

$$* \quad \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$$

Ex. let $J = \{1, 2\}$, $I = \{1, 2, 3\}$

Then $*$ says

$$A_1 \cup A_2 \subseteq A_1 \cup A_2 \cup A_3$$

This looks true.

Proof of *

Suppose $x \in \bigcup_{\alpha \in J} A_\alpha$. Then $x \in A_\beta$ for some $\beta \in J$. Since $J \subseteq I$, we also have $\beta \in I$. Since $x \in A_\beta$, we have

$$x \in \bigcup_{\alpha \in I} A_\alpha .$$

Thus every member of LHS of * is also a member of RHS of *. Therefore LHS $\subseteq$ RHS, i.e.

$$\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha .$$

What is a Proof?

It is a finite sequence of statements, starting with the hypothesis ($\emptyset \neq J \subseteq I$) and ending with the conclusion ($\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$).

Each statement follows logically from some previous statement(s) by some

(1) definition

(2) inference rule

(3) previously proved fact.

what do the expressions

$$\bigcup_{\alpha \in I} A_\alpha$$

and

$$\bigcap_{\alpha \in I} A_\alpha$$

mean when $I = \emptyset$?

<u>Answer</u>

- $\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset$

- $\bigcap_{\alpha \in \emptyset} A_\alpha$ requires a universal set $\mathcal{U}$ which contains all A_α. In this case

$$\bigcap_{\alpha \in \emptyset} A_\alpha = \mathcal{U}$$

we conclude that * is true even when $J = \emptyset$.

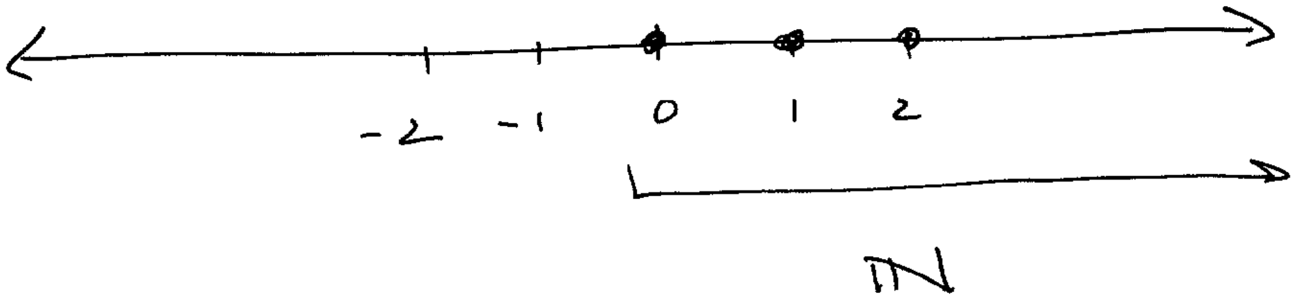
$$* \quad \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$$

whenever $J \subseteq I$.

(1.9) Number Systems

we know:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$



$\mathbb{N}$ has a special property:

<u>well ordering property of $\mathbb{N}$</u>

Any non-empty subset of $\mathbb{N}$ contains a least element.

In other words: if $A \subseteq \mathbb{N}$ and $A \neq \emptyset$, then there exists $x_0 \in A$ such that $x_0 \leq x$ for all $x \in A$.

This is such a basic fact, that its not a Theorem. This is an Axiom, a statement assumed without proof.

The study of $\mathbb{N}$, $\mathbb{Z}$ and their subsets, is called <u>number theory</u>.

<u>Theorem</u> (Division Algorithm)

Given $a, b \in \mathbb{Z}$ with $b > 0$, there exist unique $q, r \in \mathbb{Z}$ such that

$$\underbrace{a}_{\text{dividend}} = \underbrace{q}_{\text{quotient}}\,\underbrace{b}_{\text{divisor}} + \underbrace{r}_{\text{remainder}} \quad \text{and} \quad 0 \le r < b$$

Integer division (unlike real division) is an operation with 2 inputs & 2 outputs

$$a, b \longrightarrow \boxed{\text{int div}} \longrightarrow \begin{cases} q = a \ \underline{\text{div}}\ b = a/b \\ r = a \ \underline{\text{mod}}\ b = \underbrace{a \% b}_{\text{C, C++, Java}} \end{cases}$$

There are two things to prove

(1) existence

(2) uniqueness

Proof of (1)

Given $a, b \in \mathbb{Z}$ with $b > 0$. Let

$$A = \{ a - xb \mid x \in \mathbb{Z} \text{ and } 0 \leq a - xb \leq a \}$$

observe $A \subseteq \mathbb{N}$ and $A \neq \phi$ $(a \in A)$.

By the W.O.P. of $\mathbb{N}$, A contains a least element. Call it r.

Thus

$$r = a - qb \quad \text{for some } q \in \mathbb{Z}$$

and

$$r \geq 0$$

must show $0 \overset{\checkmark}{\le} r \overset{?}{<} b$.

Suppose $r \geq b$. Then we would have

$$0 \leq r - b = (a - qb) - b = a - (q+1)b < a - qb.$$

This is impossible since $r = a - qb$ is the <u>least</u> element $a - xb \in A$.

Therefore $r \geq b$ is false!

Therefore $r < b$.

We've shown q, r exist with

$$a = qb + r \quad \text{and} \quad 0 \leq r < b.$$

$\blacksquare$

Exercise:

Prove (2) uniqueness, i.e.

suppose both

$$a = q_1 b + r_1 \quad , \quad 0 \le r_1 < b$$

and

$$a = q_2 b + r_2 \quad , \quad 0 \le r_2 < b \,.$$

Then show that $q_1 = q_2$ and $r_1 = r_2$.

John von Neuman invented the modern method for constructing the natural numbers: 0, 1, 2, 3, ... out of sets.

one basic operation on sets

<u>successor</u> : $S^+ = S \cup \{S\}$

$$0 = \emptyset$$
$$1 = 0^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$$
$$2 = 1^+ = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\}$$
$$3 = 2^+ = 2 \cup \{2\} = \{0, 1\} \cup \{2\} = \{0, 1, 2\}$$
$$\vdots$$
$$n = \{0, 1, 2, \ldots, n-1\}$$
$$\vdots$$

so $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$.

Its possible to prove that $\mathbb{N}$ satisfies the W.O.P.

(1.10) Russel's Paradox

Can there be a set that has itself as a member? say

$$A = \{A\} = \{\{A\}\}$$
$$= \{\{\{A\}\}\}$$
$$\vdots$$
$$= \{\{\{\cdots\}\}\}$$

Our naieve definition poses no obstacle to this! In fact we can form:

$$\mathcal{S} = \{x \mid x \text{ is a set}\}$$

The set of all sets.

observe $\mathcal{S} \in \mathcal{S}$. Also note $\phi \notin \phi$. Evidently some sets are members of themselves, and some are not.

Define

$$\mathcal{R} = \{ A \in \mathcal{S} \mid A \notin A \}.$$

These are the "normal" sets.

Clearly either $\mathcal{R} \in \mathcal{R}$ or $\mathcal{R} \notin \mathcal{R}$.

- If $\mathcal{R} \in \mathcal{R}$, then $\mathcal{R}$ satisfies $\mathcal{R} \notin \mathcal{R}$.
- If $\mathcal{R} \notin \mathcal{R}$, then $\mathcal{R}$ satisfies its own defining property, so $\mathcal{R} \in \mathcal{R}$.

next time start

chap 2 : logic.

(2.1) statements

Defn

A statement (or proposition) is a sentence that can be assigned a truth value (true or false).