CSE 16 6-23-21

(1.3) Subsets

Defn

we say a set A is a subset of a set B iff every element of A is also an element of B.

notation: $A \subseteq B$

we say A is a proper subset of B iff $A \subseteq B$ but $A \neq B$.

notation: $A \subsetneq B$

note: for any set S:

- $S \subseteq S$

and
- $\phi \subseteq S$

Remarks

- some books use $\subset$ for subset, others use it for proper subset. we won't use $\subset$.

- 'contains' and 'in'

  - $x \in S$: x is in S, S contains x.

  - $A \subseteq S$: A is in S, S contains A.

(1.4) Power set of a set.

Defn

The set of all subsets of a set S is called the Power set of S.

notation: $\mathbb{P}(S)$.

Ex. $S = \{1, 2, 3\}$, $|S| = 3$

$$\mathbb{P}(S) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$$

observe:

$$|\mathbb{P}(S)| = 8 = 2^3 = 2^{|S|}$$

<u>Theorem</u>

If S is a finite set, then so is $\mathbb{P}(S)$ and

$$|\mathbb{P}(S)| = 2^{|S|}$$

<u>notation</u> some books write 2^S for $\mathbb{P}(S)$, so $|2^S| = 2^{|S|}$.

<u>Ex.</u> $\mathbb{P}(\emptyset) = \{\emptyset\}$ <u>note:</u> $2^0 = 1$

<u>Ex.</u> $\mathbb{P}(\{1\}) = \{\emptyset, \{1\}\}$

<u>Ex.</u> $\mathbb{P}(\mathbb{P}(\emptyset)) = \mathbb{P}(\{\emptyset\})$

$$= \{\emptyset, \{\emptyset\}\}$$

Ex. $\mathbb{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Ex.

$$\mathbb{P}(\mathbb{P}(\mathbb{P}(\emptyset))) = \mathbb{P}(\{\emptyset, \{\emptyset\}\})$$

$$= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

note: $2^2 = 4$

Exercise

find an expression for

$$\Big|\underbrace{\mathbb{P}(\mathbb{P}(\mathbb{P}(\cdots\cdots\mathbb{P}}(\emptyset)\cdots\cdots)))\Big|$$

number of $\mathbb{P}$'s is n.

(1.5) set operations

Defn

let A, B be sets.

inclusive or

- **union** : $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- **intersection** : $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- **difference** : $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Ex. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$

- $A \cup B = \{1, 2, 3, 4, 5\}$
- $A \cap B = \{3, 4\}$
- $A - B = \{1, 2\}$
- $B - A = \{5\}$

observe:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

However $A - B \neq B - A$ in general.
$A - B = B - A$ is only true if $A = B$,
in which case both sides are ϕ.

Defn
we say A, B are disjoint iff

$$A \cap B = \phi$$

Observe:

- $(A - B) \cap B = \phi$
- $(B - A) \cap A = \phi$
- $A - B \subseteq A$
- $B - A \subseteq B$

(1.6) complements

Suppose all sets under discussion belong to some universal set $\mathcal{U}$.

Defn

The complement of A (relative to $\mathcal{U}$) is

$$\bar{A} = \mathcal{U} - A$$
$$= \{x \in \mathcal{U} \mid x \notin A\}$$

Ex. $\mathcal{U} = \mathbb{N}$

$$A = \{x \in \mathbb{N} \mid x \geq 8\}$$
$$= \{8, 9, 10, \ldots\ldots\}$$
$$\bar{A} = \mathbb{N} - A = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

Ex. $U = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$A = \{x \in \mathbb{Z} \mid x \geq 8\}$

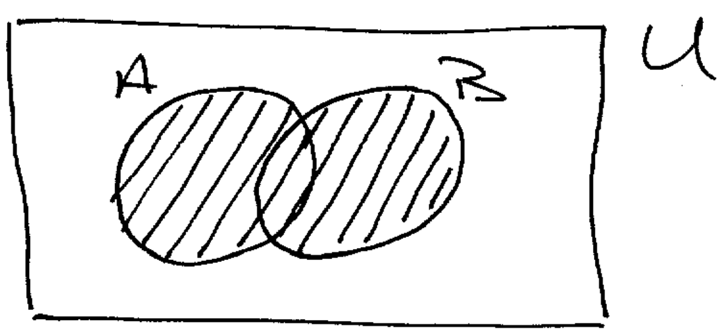
$= \{8, 9, 10, \ldots\}$

$\bar{A} = \mathbb{Z} - A$

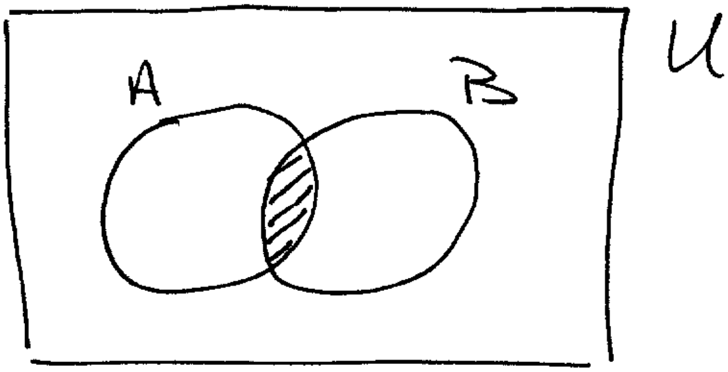
$= \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$

(1.7) Venn Diagrams

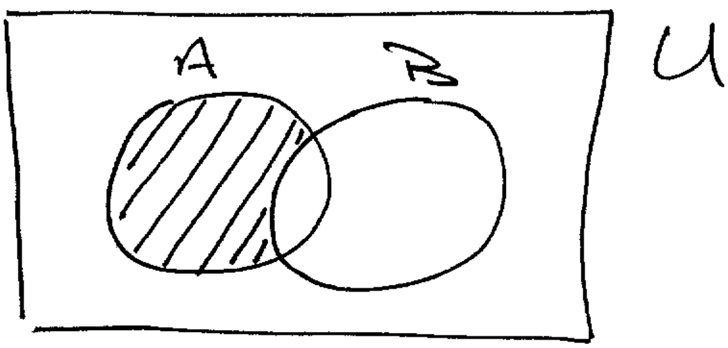
examples



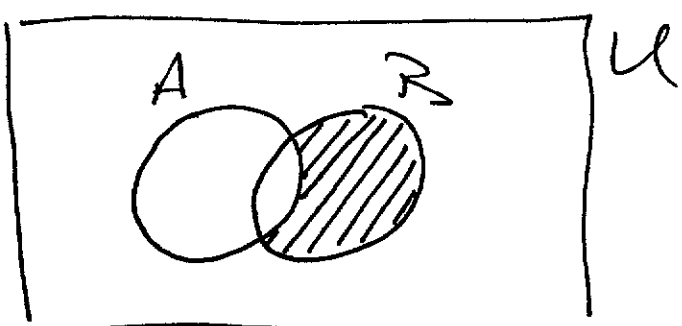
$A \cup B$



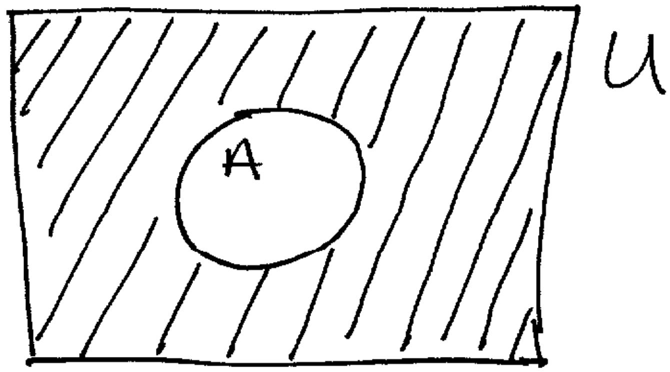
$A \cap B$



$A - B$

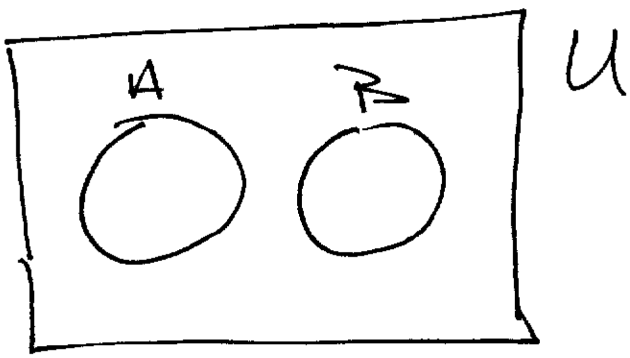


$B - A$



$\overline{A}$

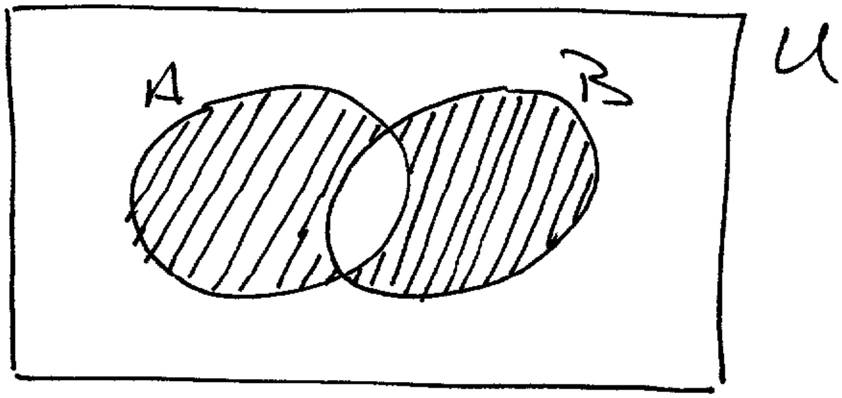
Disjoint sets:



$A \cap B = \phi$

new operation: <u>Symmetric difference</u>

$$A \oplus B = (A - B) \cup (B - A)$$

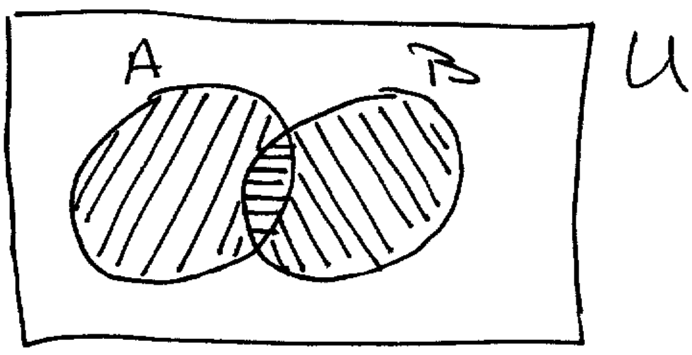


$A \oplus B$

we can see that:

$$A \oplus B = (A \cup B) - (A \cap B)$$

Also:

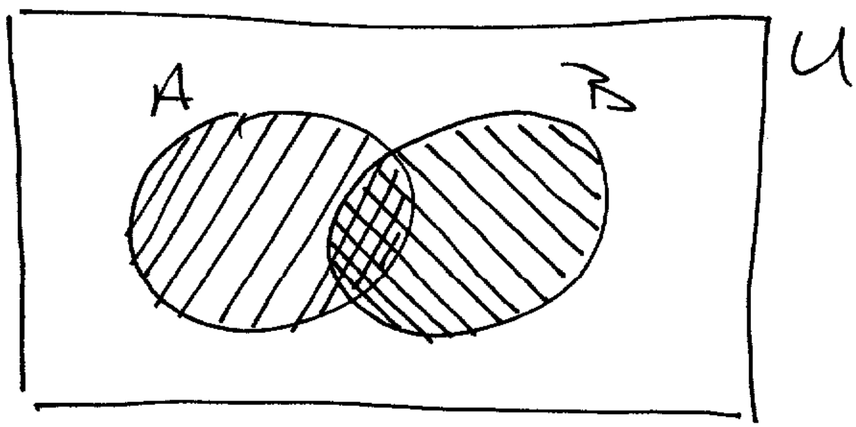


$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

Theorem

If A, B are finite sets, then so are $A \cup B$ and $A \cap B$, also

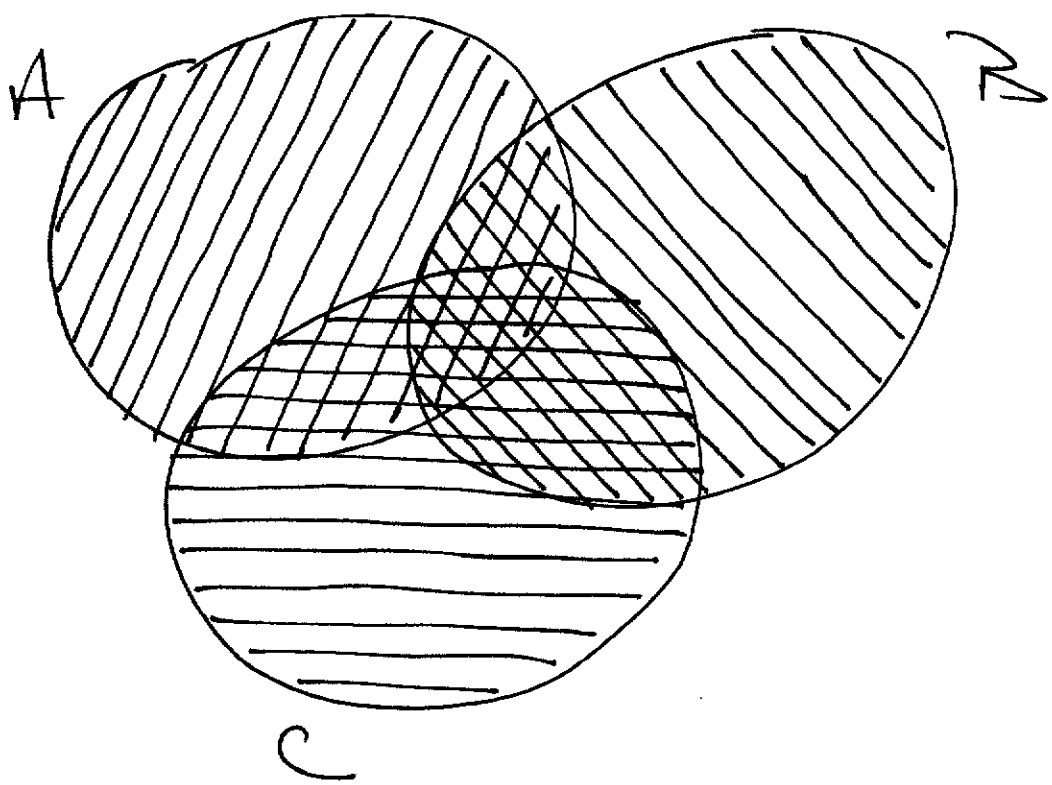
$$|A \cup B| = |A| + |B| - |A \cap B|$$



Why? $|A| + |B|$ counts elements in $A \cap B$ twice, so must subtract them once to count all elements in $A \cup B$.

This is a special case of the <u>principle of inclusion-exclusion</u> (PIE)

<u>3-set version</u>



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise

Write the 4-set version, Draw a Venn diagram.

(1.8) Indexed Sets

Let $A_1, A_2, A_3, \ldots\ldots$ be a (possibly infinite) list of sets.

notation

- $$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$
  $$= \{x \mid x \in A_i \text{ for } \underbrace{\text{at least one}}_{\text{some}} \ 1 \le i \le n\}$$
- $$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots\cdots$$
  $$= \{x \mid x \in A_i \text{ for } \underbrace{\text{at least one}}_{\text{some}} \ i \ge 1\}$$

- $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots\cdots \cap A_n$

$$= \{x \mid x \in A_i \text{ for } \underline{\text{all}}\ 1 \le i \le n\}$$

- $\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots\cdots\cdots$

$$= \{x \mid x \in A_i \text{ for } \underline{\text{all}}\ i \ge 1\}$$

we can pick out specific indicies

<u>Ex.</u>

$$\bigcup_{i \in \{2,7,11\}} A_i = A_2 \cup A_7 \cup A_{11}$$

$$\bigcap_{i \in \{2,7,11\}} A_i = A_2 \cap A_7 \cap A_{11}$$

In general suppose we have a set A_α for every $\alpha \in I$ (where I is any <u>index set</u>, finite or infinite.)

$$\bigcup_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for } \underline{\text{some}}\ \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for } \underline{\text{all}}\ \alpha \in I\}$$

⋮

more to come

⋮