CSE 16
Summer 2020
Quiz 6 **Solutions**

1. (10 Points) Let F_n be the n^{th} Fibonacci number. Use induction to prove

$$\forall n \geq 1: \quad \sum_{k=1}^{n} F_{2k-1} = F_{2n}$$

Proof: (Using weak induction form IIa.)

I. If $n = 1$, we have $\sum_{k=1}^{1} F_{2k-1} = F_{2\cdot 1-1} = F_1 = 1 = F_2 = F_{2\cdot 1}$.

II. Let $n \geq 1$. Assume $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$. We must show $\sum_{k=1}^{n+1} F_{2k-1} = F_{2(n+1)}$. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} F_{2k-1} &= \left(\sum_{k=1}^{n} F_{2k-1}\right) + F_{2(n+1)-1} \\
&= F_{2n} + F_{2n+1} && \text{(by the induction hypothesis)} \\
&= F_{2n+2} && \text{(by the Fibonacci recurrence)} \\
&= F_{2(n+1)},
\end{aligned}$$

as required. The result follows for all $n \geq 1$ by the 1st PMI. ■

Alternate Proof: (Using weak induction form IIb.)

I. Same as above

II. Let $n > 1$. Assume $\sum_{k=1}^{n-1} F_{2k-1} = F_{2(n-1)} = F_{2n-2}$. We must show $\sum_{k=1}^{n} F_{2k-1} = F_{2n}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} F_{2k-1} &= \left(\sum_{k=1}^{n-1} F_{2k-1}\right) + F_{2n-1} \\
&= F_{2n-2} + F_{2n-1} && \text{(by the induction hypothesis)} \\
&= F_{2n} && \text{(by the Fibonacci recurrence),}
\end{aligned}$$

as required. The result follows for all $n \geq 1$ by the 1st PMI. ■

2. (20 Points) Let x, y and n denote integers. Use induction to prove that

$$\forall n \geq 5: \ \exists x, y \geq 0 \ \text{ such that } \ n = 2x + 3y$$

(Hint: use strong induction form IId, with two base cases: $n = 5$ and $n = 6$.)

Proof:

I. We have $5 = 2 \cdot 1 + 3 \cdot 1$ and $6 = 2 \cdot 0 + 3 \cdot 2$, showing that the result is true when $n = 5$ and when $n = 6$.

II. Let $n > 6$ be chosen arbitrarily. Assume for all k in the range $5 \leq k < n$ that

$$\exists x_k, y_k \geq 0 \ \text{ such that } \ k = 2x_k + 3y_k.$$

We must show that

$$\exists x, y \geq 0 \ \text{ such that } \ n = 2x + 3y.$$

Observe $n > 6 \ \Rightarrow \ n \geq 7 \ \Rightarrow \ n - 2 \geq 5$. Therefore, by the induction hypothesis, there exist integers $x', y' \geq 0$ such that $n - 2 = 2x' + 3y'$. Let $x = x' + 1$ and $y = y'$. Then

$$\begin{aligned} n &= (n-2) + 2 \\ &= (2x' + 3y') + 2 \\ &= 2(x' + 1) + 3y' \\ &= 2x + 3y, \end{aligned}$$

as required. The result follows for all $n \geq 5$ by the 2nd PMI. ∎

3. (20 Points) Let $A = \{1, 2, 3, 4\}$ and suppose that R is an equivalence relation on A. Suppose also that $1R2$, $1R3$ and $1\not R4$.

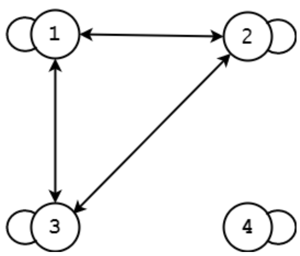
a. (10 Points) Write R as a set of ordered pairs, **or** draw its digraph representation.

Solution:

<u>Ordered pairs:</u>

$R = \{ (1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

<u>Digraph:</u>



b. (10 Points) Determine the equivalence classes of R.

Solution:

$[1] = [2] = [3] = \{1, 2, 3\}$

$[4] = \{4\}$