**CSE 16**
**Summer 2020**
**Quiz 5** **Solutions**

1. (25 Points) Let $n \in \mathbb{N}$. Prove that $n^2 \not\equiv 2 \pmod 3$.

**Proof:**
Every $n \in \mathbb{N}$ has one of three possible remainders upon division by 3 (0, 1 or 2). Therefore each $n \in \mathbb{N}$ satisfies exactly one of $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. We have three cases to consider.

<u>Case 0:</u> $n \equiv 0 \pmod 3$
In this case $n = 3k$ (for some $k \in \mathbb{Z}$), so $n^2 = 9k^2 = 3(3k^2)$, whence $3|n^2$, and $n^2 \equiv 0 \pmod 3$.

<u>Case 1:</u> $n \equiv 1 \pmod 3$
In this case $n = 3k + 1$ (for some $k \in \mathbb{Z}$), therefore $n^2 = (3k+1)^2 = 3(3k^2 + 2k) + 1$, showing that $n^2 \equiv 1 \pmod 3$.

<u>Case 2</u>: $n \equiv 2 \pmod 3$
In this case $n = 3k + 2$ (for some $k \in \mathbb{Z}$), and $n^2 = 3(3k^2 + 4k + 1) + 1$. Again $n^2 \equiv 1 \pmod 3$.

In *no case* do we have $n^2 \equiv 2 \pmod 3$. Since these cases exhaust all possibilities, we have established that $n^2 \not\equiv 2 \pmod 3$ for any $n \in \mathbb{N}$. ■

2. (25 Points) Use induction to prove

$$\forall n \geq 1: \quad \sum_{k=1}^{n} k(k+2) = \frac{n(n+1)(2n+7)}{6}$$

**Proof:**

I. If $n = 1$, we have $\sum_{k=1}^{1} k(k+2) = \frac{1\cdot(1+1)(2\cdot 1+7)}{6}$, i.e. $1 \cdot (1+3) = \frac{2\cdot 9}{6}$, i.e $3 = 3$, which is true.

II. Let $n \geq 1$ be arbitrary. Assume (for this $n$) that

$$\sum_{k=1}^{n} k(k+2) = \frac{n(n+1)(2n+7)}{6}.$$

We must show that

$$\sum_{k=1}^{n+1} k(k+2) = \frac{(n+1)(n+2)(2n+9)}{6}.$$

We have

$$\begin{aligned}
\sum_{k=1}^{n+1} k(k+2) &= \left(\sum_{k=1}^{n} k(k+2)\right) + (n+1)(n+3) \\
&= \left(\frac{n(n+1)(2n+7)}{6}\right) + (n+1)(n+3) \quad \text{(by the induction hypothesis)} \\
&= (n+1)\left(\frac{n(2n+7)}{6} + \frac{6(n+3)}{6}\right) \\
&= (n+1)\left(\frac{2n^2+7n+6n+18}{6}\right) \\
&= \frac{(n+1)(2n^2+13n+18)}{6} \\
&= \frac{(n+1)(n+2)(2n+9)}{6}.
\end{aligned}$$

The result follows for all $n \geq 1$ by the first principle of mathematical induction. ■