hw 8
complete solutions

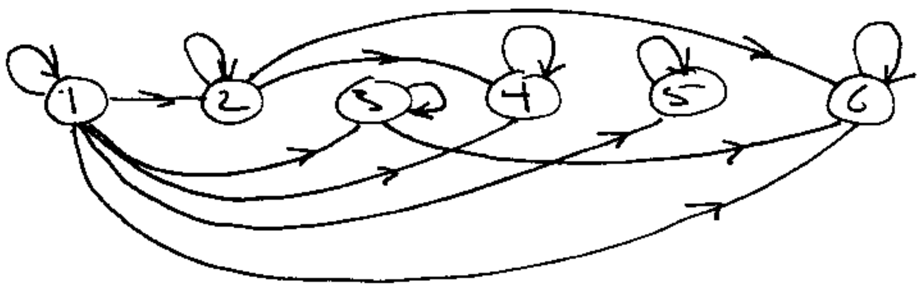
11.1 #2

$A = \{1, 2, 3, 4, 5, 6\}$

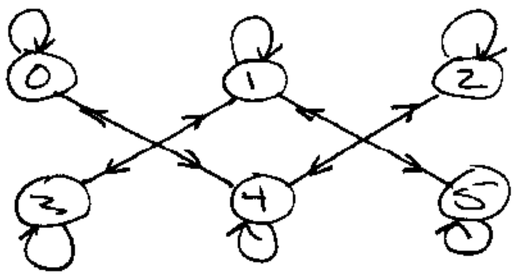
$R = \{(x, y) \in A \times A \mid x \mid y\}$

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$

$(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$



11.1 #4



$A = \{0, 1, 2, 3, 4, 5\}$

$R = \{(0,0), (0,4), (1,1), (1,3), (1,5), (2,2), (2,4),$

$(3,3), (3,1), (4,0), (4,2), (4,4), (5,1), (5,5)\}$

11.1 #6

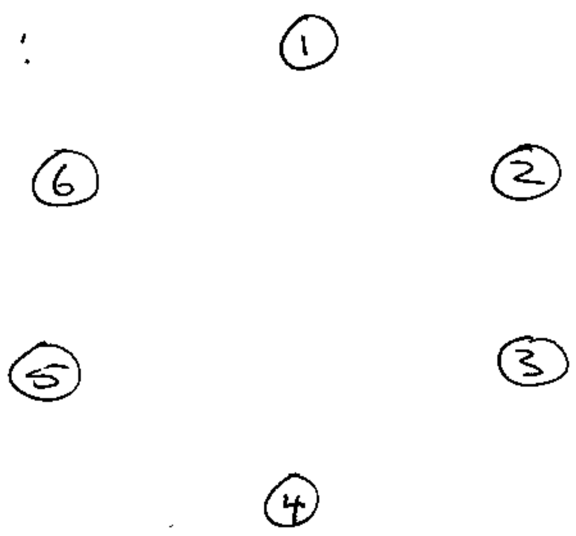
$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \pmod 5\}$$

11.1 #8

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$R = \emptyset$$

Digraph:



11.1 #10

$R = (\mathbb{R}\times\mathbb{R}) - \{(x,x) \mid x\in\mathbb{R}\}$

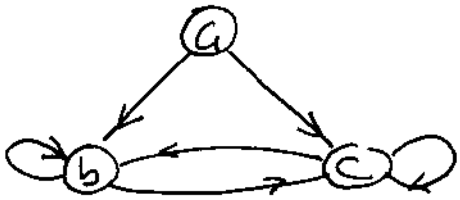
R is the "not equal" relation on the real numbers, i.e. $\neq$.

This is because $\{(x,x)\mid x\in\mathbb{R}\}$ is the "equals" relation, and R is its complement in $\mathbb{R}\times\mathbb{R}$.

11.2 #2

$A = \{a, b, c,\}$

$R = \{(a,b), (a,c), (c,c), (b,b), (c,b), (b,c)\}$



Reflexive: NO, since $a \not R a$

symmetric: NO, since aRb but $b \not R a$

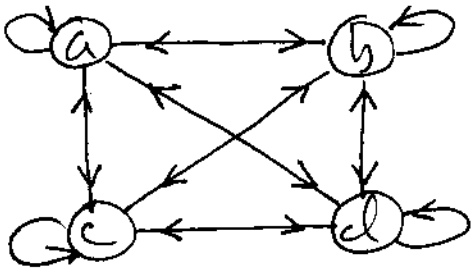
transitive: YES

11.2 # 4

$A = \{a, b, c, d\}$

$R = \{ (a,a), (b,b), (c,c), (d,d), (a,b),$
$(b,a), (a,c), (c,a), (a,d), (d,a)$
$(b,c), (c,b), (b,d), (d,b), (c,d), (d,c) \}$

note $R = A \times A$.



Reflexive: YES
Symmetric: YES
Transitive: YES

11.2 #6

$$R = \{(x,x) \mid x \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

Reflexive: YES
Symmetric: YES
Transitive: YES

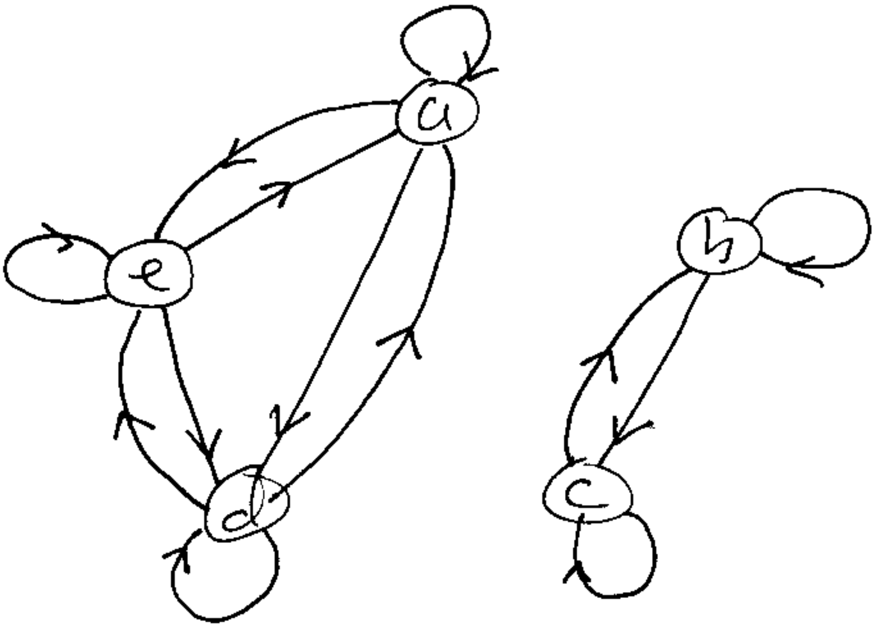
R is the "equals" relation on $\mathbb{Z}$, i.e. $=$.

11.3 #2

$A = \{a, b, c, d, e\}$

$R \subseteq A \times A$ an equiv. rel.

$\boxed{\text{2 equiv. classes.}}$



$$\boxed{aRd,\ bRc,\ eRd}$$

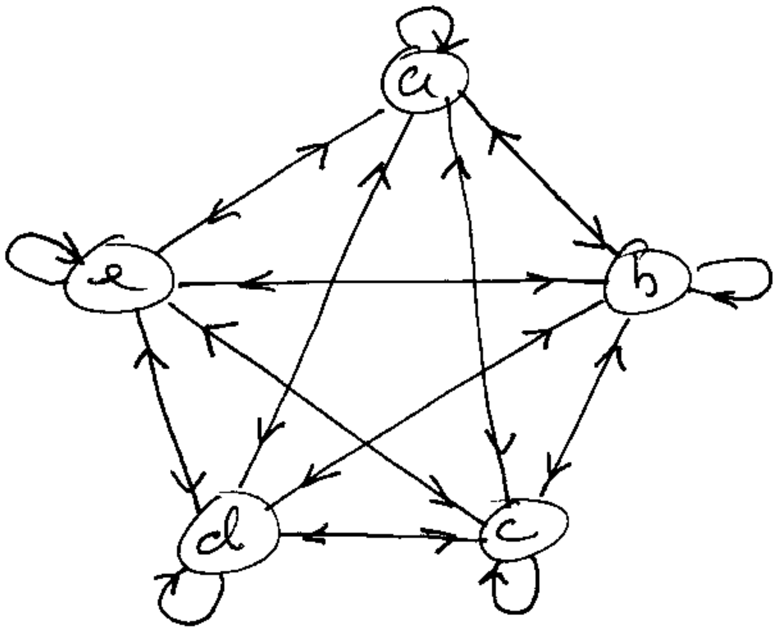
$$R = \{(a,a), \underline{(a,d)}, (a,e), (b,b), \underline{(b,c)}, (c,b), (c,c),$$
$$(d,d), (d,a), (d,e), (e,a), \underline{(e,d)}, (e,e)\}$$

11.3 #4

$A = \{a, b, c, d, e\}$

$R \subseteq A \times A$ an eq. rel.

Assume aRd, bRc, eRa, cRe

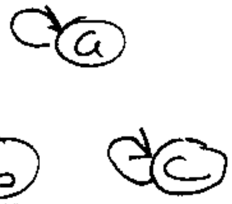


$\boxed{\text{1 eq. class}}$

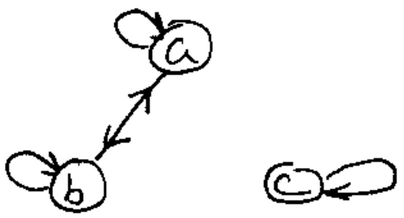
11.3 #6

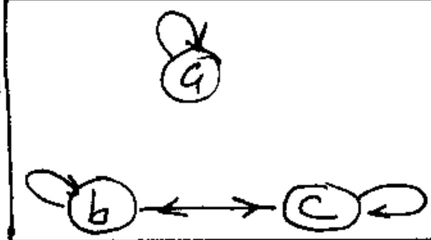
$A = \{a, b, c\}$

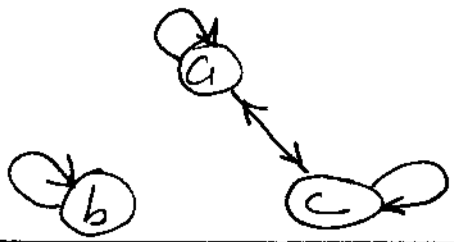
describe all five eq. relations on A.

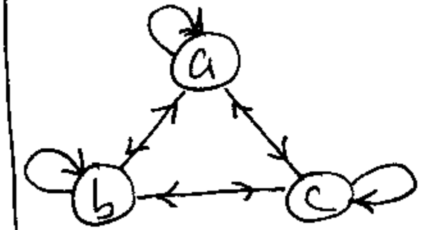
1.) 

this is the $=$ relation on A.

2.) 

3.) 

4.) 

5.) 

this is the "full" relation on A, i.e. $A \times A$.

11.3 #8

On $\mathbb{Z}$: xRy iff x^2+y^2 even.

→ Prove R is an eq. rel.

Pf.

- Ref. $x^2+x^2=2x^2$ is even $\therefore xRx$
- Symm. x^2+y^2 even $\Rightarrow y^2+x^2$ even $\therefore xRy \Rightarrow yRx$.
- trans. x^2+y^2 even $\wedge$ y^2+z^2 even

$$\Rightarrow (x^2+y^2)+(y^2+z^2)=(x^2+z^2)+2y^2 \text{ even}$$
$$\Rightarrow (x^2+z^2)+2y^2=2k$$
$$\Rightarrow x^2+z^2=2(k-y^2) \text{ even}$$
$$\therefore xRy \wedge yRz \Rightarrow xRz.$$

→ what are eq. classes ?

Recall x & x^2 always have same parity.

$\therefore$ xRy iff x^2+y^2 even
iff x^2, y^2 have same parity $\begin{cases} e+e=e \\ o+o=e \end{cases}$
iff x, y have same parity.

$\therefore$ eq classes are

$E=\{\text{even \#s}\}=\{x \mid x=2k,\ k\in\mathbb{Z}\}$

$O=\{\text{odd \#s}\}=\{x \mid x=2k+1,\ k\in\mathbb{Z}\}$

note. $R=(\equiv_2)$.

11.4 # 2

List all Partitions of $\{a,b,c\}$

1.) $\{\{a\},\{b\},\{c\}\}$

2.) $\{\{a,b\},\{c\}\}$

3.) $\{\{a\},\{b,c\}\}$

4.) $\{\{a,c\},\{b\}\}$

5.) $\{\{a,b,c\}\}$

note

These are the sets of equivalence classes of the 5 equivalence relations in 11.3 #6.

more exercises from chapter 10

ch. 10 #25

Prove: $\forall n \geq 1: \sum_{k=1}^{n} F_k = F_{n+2} - 1$

Proof:

I. $n=1$: $\sum_{k=1}^{1} F_k = F_3 - 1$

i.e. $F_1 = F_3 - 1$

i.e. $1 = 2 - 1$ ✓

IIa. let $n \geq 1$. Assume

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1 \qquad \text{(ind. hyp.)}$$

show

$$\sum_{k=1}^{n+1} F_k = F_{n+3} - 1 \qquad \text{(ind. concl.)}$$

so
$$\begin{aligned}
\sum_{k=1}^{n+1} F_k &= \left(\sum_{k=1}^{n} F_k\right) + F_{n+1} \\
&= (F_{n+2} - 1) + F_{n+1} && \text{\{by the ind. hyp.} \\
&= (F_{n+1} + F_{n+2}) - 1 \\
&= F_{n+3} - 1 && \text{\{by the Fib. recurrence}
\end{aligned}$$

Result follows by PMI. ∎

Ch. 10 #29

$$\forall n \geq 0 \ : \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

Convention: $\binom{a}{b} = 0$ if $a < b$ or $b < 0$.

Proof.
We use strong ind. form IId
with 2 base cases

I. $n = 0$: $\binom{0}{0} = F_{0+1}$, $1 = 1$ ✓

$n = 1$: $\binom{1}{0} + \cancel{\binom{0}{1}} = F_{1+1}$, $1 = 1$ ✓

IId. $\forall n > 1 : P(0) \wedge \dots \wedge P(n-1) \Rightarrow P(n)$.

Assume for all l s.t. $0 \leq l < n$:

$$\sum_{k=0}^{l} \binom{l-k}{k} = F_{l+1}$$

We must show

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

so

$$\sum_{k=0}^{n} \binom{n-k}{k} = \sum_{k=0}^{n} \left(\binom{n-k-1}{k-1} + \binom{n-k-1}{k} \right)$$

(by Pascal's identity)

$$= \sum_{k=0}^{n} \binom{n-k-1}{k-1} + \sum_{k=0}^{n} \binom{n-k-1}{k}$$

$$= \sum_{k=-1}^{n-1} \binom{n-(k+1)-1}{k} + \sum_{k=0}^{n} \binom{n-k-1}{k}$$

(replace k by k+1 in 1st sum)

$$= \sum_{k=-1}^{n-1} \binom{(n-2)-k}{k} + \sum_{k=0}^{n} \binom{(n-1)-k}{k}$$

$$= \cancel{\binom{n-3}{-1}} + \sum_{k=0}^{n-2} \binom{(n-2)-k}{k} + \cancel{\binom{-1}{n-1}} + \sum_{k=0}^{n-1} \binom{(n-1)-k}{k} + \cancel{\binom{-1}{n}}$$

$$= F_{(n-2)+1} + F_{(n-1)+1}$$

$\left\{ \begin{array}{l} \text{by the ind.} \\ \text{hyp. } l = n-2 \\ \text{and } l = n-1 \end{array} \right.$

$$= F_{n-1} + F_n = F_{n+1}.$$

Result holds by 2nd. PMI. ∎