CSE 16
Homework Assignment 7
Solutions to Selected Problems

Chapter 9 (page 179): # 26
Prove or disprove the statement: for any sets A, B and C, if $A = B - C$, then $B = A \cup C$.

Solution: The statement is false.

Disproof: We must show that there exist sets A, B and C for which $A = B - C$, but $B \neq A \cup C$.

Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{2, 3\}$. Then

$$B - C = \{1, 2\} - \{2, 3\} = \{1\} = A$$

and

$$A \cup C = \{1\} \cup \{2, 3\} = \{1, 2, 3\} \neq B.$$

■

Chapter 10 (page 195): # 12
Prove (by induction) that $\forall n \geq 0$: $9|(4^{3n} + 8)$.

Proof:
Let $P(n)$ be the assertion $9|(4^{3n} + 8)$. We use weak induction form IIa.

I. If $n = 0$, then $4^{3n} + 8 = 4^0 + 8 = 1 + 8 = 9$, hence $9|(4^{3\cdot 0} + 8)$, showing that $P(0)$ is true.

II. Let $n \geq 0$ be chosen arbitrarily. Assume, for this n, that $9|(4^{3n} + 8)$. Therefore $4^{3n} + 8 = 9k$ for some $k \in \mathbb{Z}$. We must show that $9|(4^{3(n+1)} + 8)$. Observe

$$\begin{aligned} 4^{3(n+1)} + 8 &= 4^{3n+3} + 8 \\ &= 4^3 \cdot 4^{3n} + 8 \\ &= (63 + 1) \cdot 4^{3n} + 8 \\ &= 63 \cdot 4^{3n} + (4^{3n} + 8) \\ &= 9(7 \cdot 4^{3n}) + 9k && \text{By the induction hypothesis} \\ &= 9(7 \cdot 4^{3n} + k), \end{aligned}$$

showing that $9|(4^{3(n+1)} + 8)$ as required.

The result follows for all $n \geq 0$ by the 1st Principle of Mathematical Induction. ■

Chapter 10 (page 196): # 24

Prove (by induction) that $\forall n \geq 1$: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

Proof:

Let $P(n)$ be the sentence $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. We use weak induction form IIb.

I. $P(1)$ says $\sum_{k=1}^{1} k\binom{1}{k} = 1 \cdot 2^{1-1}$, i.e. $1 \cdot \binom{1}{1} = 1 \cdot 2^0$, i.e. $1 = 1$, which is true.

II. Let $n \geq 1$ be arbitrary. Assume, for this n, that $\sum_{k=1}^{n-1} k\binom{n-1}{k} = (n-1)2^{n-2}$. We must show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. We have

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] \qquad \text{by Pascal's identity}$$

$$= \sum_{k=1}^{n} k\binom{n-1}{k-1} + \sum_{k=1}^{n} k\binom{n-1}{k}$$

$$= \sum_{k=0}^{n-1} (k+1)\binom{n-1}{k} + \sum_{k=1}^{n} k\binom{n-1}{k} \qquad \text{replacing } k \text{ by } k+1 \text{ in the } 1^{\text{st}} \text{ sum}$$

$$= \sum_{k=0}^{n-1} \left[k\binom{n-1}{k} + \binom{n-1}{k}\right] + \sum_{k=1}^{n} k\binom{n-1}{k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} + \sum_{k=0}^{n-1} k\binom{n-1}{k} + \sum_{k=1}^{n} k\binom{n-1}{k}$$

$$= 2^{n-1} + 0 \cdot \binom{n-1}{0} + 2 \cdot \sum_{k=1}^{n-1} k\binom{n-1}{k} + n\binom{n-1}{n}$$

$$= 2^{n-1} + 0 + 2 \cdot (n-1)2^{n-2} + 0 \qquad \text{by the induction hypothesis}$$

$$= 2^{n-1} + (n-1)2^{n-1}$$

$$= n2^{n-1},$$

as required. The result follows for all $n \geq 1$ by the 1^{st} PMI. ■

Chapter 10 (page 196): # 26
Prove (by induction) that $\forall n \geq 1$: $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, where F_n is the n^{th} Fibonacci number.

Proof:
Let $P(n)$ be the sentence $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We use weak induction form IIa.

I. $P(1)$ says $\sum_{k=1}^{1} F_k^2 = F_1 F_{1+1}$, i.e. $F_1^2 = F_1 F_2$, i.e. $1^2 = 1 \cdot 1$, or $1 = 1$, which is true.

II. Let $n \geq 1$. Assume $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$. We must show $\sum_{k=1}^{n+1} F_k^2 = F_{n+1} F_{n+2}$. We have

$$
\begin{aligned}
\sum_{k=1}^{n+1} F_k^2 &= \left(\sum_{k=1}^{n} F_k^2\right) + F_{n+1}^2 \\
&= (F_n F_{n+1}) + F_{n+1}^2 && \text{by the induction hypothesis} \\
&= F_{n+1}(F_n + F_{n+1}) \\
&= F_{n+1} F_{n+2} && \text{by the Fibonacci recurrence}
\end{aligned}
$$

It follows that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ for all $n \geq 1$ by the 1st PMI. ■