CSE 16
Homework Assignment 5
Solutions to Selected Problems

<u>Section 3.10: 6</u>
Show that $\binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$.

Proof:
Let S be a set with $|S| = 3n$, and let A, B and C be subsets of S satisfying $A \cap B = A \cap C = B \cap C = \emptyset$, and $|A| = |B| = |C| = n$. The left hand side of the above formula, by its very definition, counts the number of 3-element subsets of S, i.e. the number of $\{x, y, z\} \subseteq S$, with x, y, z distinct. We count these same subsets in another way to obtain the right hand side.

Partition the 3-element subsets of S into 3 mutually exclusive classes.

I. Choose $\{x, y, z\} \subseteq A$: #ways $= \binom{n}{3}$
Choose $\{x, y, z\} \subseteq B$: #ways $= \binom{n}{3}$
Choose $\{x, y, z\} \subseteq C$: #ways $= \binom{n}{3}$
Total number of sets of this type: $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3\binom{n}{3}$

II. Choose $x \in A$ and $\{y, z\} \subseteq B$: #ways $= n\binom{n}{2}$
Choose $x \in A$ and $\{y, z\} \subseteq C$: #ways $= n\binom{n}{2}$
Choose $x \in B$ and $\{y, z\} \subseteq A$: #ways $= n\binom{n}{2}$
Choose $x \in B$ and $\{y, z\} \subseteq C$: #ways $= n\binom{n}{2}$
Choose $x \in C$ and $\{y, z\} \subseteq A$: #ways $= n\binom{n}{2}$
Choose $x \in C$ and $\{y, z\} \subseteq B$: #ways $= n\binom{n}{2}$
Total number of sets of this type: $6n\binom{n}{2}$

III. Choose $x \in A$, $y \in B$ and $z \in C$: #ways $= n \cdot n \cdot n = n^3$

Since every 3-element subset $\{x, y, z\} \subseteq S$ falls into exactly one of the above classes, the addition principle gives the number of such subsets as $3\binom{n}{3} + 6n\binom{n}{2} + n^3$, which proves the formula. ■

<u>Section 3.10: 10</u>
Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

Proof:
Let S be a set with $|S| = n$. Both sides of the above equation count the number of pairs (x, A) for which $x \in A \subseteq S$. (Note this is similar to the example on p.8 of the lecture notes from 7-16-20, except there is no restriction on the size of the subset A, other than $|A| \geq 1$.) Alternately, given a set of n players, count the number of teams that can be formed (of any size other than zero) with one player designated as captain.

To construct an element of $\{ (x, A) \mid x \in A \subseteq S \}$ we

(1) choose $x \in S$ (n ways), then
(2) choose $B \subseteq S - \{x\}$ (2^{n-1} ways), and let $A = \{x\} \cup B$.

By the multiplication principle, the number of ways of making these choices in succession is $n2^{n-1}$, which is the right hand side.

Another way to construct (x, A) is to first choose a non-empty subset $A \subseteq S$, then pick $x \in A$. This method splits into n mutually exclusive cases.

$|A| = 1$: #ways $= \binom{n}{1} \cdot 1$
$|A| = 2$: #ways $= \binom{n}{2} \cdot 2$
$|A| = 3$: #ways $= \binom{n}{3} \cdot 3$
$\vdots$
$\vdots$
$|A| = n$: #ways $= \binom{n}{n} \cdot n$

Since each (x, A) falls into exactly one of the above cases, the addition principle yields $\sum_{k=1}^{n} k \binom{n}{k}$ as the number of such pairs, which is the left hand side. Since both sides count the same set of objects, we conclude $\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$. ■

<u>Chapter 4 (page 126): 6</u>
Suppose $a, b, c \in \mathbb{Z}$. Show by direct proof that if $a|b$ and $a|c$, then $a|(b + c)$.

Proof:
Since $a|b$ and $a|c$ there exist integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 a$. Therefore

$$b + c = k_1 a + k_2 a = (k_1 + k_2)a.$$

Since $k_1 + k_2$ is also an integer, we have $a|(b + c)$, as required. ■

<u>Chapter 4 (page 126): 26</u>
Show by direct proof that every odd integer is the difference of two squares. (Example: $7 = 4^2 - 3^2$).

Proof:
Let $n \in \mathbb{Z}$ be odd. Then there exists $k \in \mathbb{Z}$ for which $n = 2k + 1$. We must show that there exist $a, b \in \mathbb{Z}$ such that $a^2 - b^2 = n$. Set $a = k + 1$ and $b = k$. Then a and b are integers, and

$$a^2 - b^2 = (k + 1)^2 - k = (k^2 + 2k + 1) - k^2 = 2k + 1 = n$$

as required. ■