CSE 16
Homework Assignment 4
Solutions to Selected Problems

(3.7) 2

How many 4-digit positive integers are there for which there are no repeated digits, or for which there may be repeated digits, but all digits are odd?

Solution:

Let

$$A = \{\text{4-digit positive integers with no repeated digits}\}$$

and

$$B = \{\text{4-digit positive integers, repetition allowed, all digits odd}\},$$

So that

$$A \cap B = \{\text{4-digit positive integers, no repeated digits, all digits odd}\}.$$

Then $|A| = 9 \cdot 9 \cdot 8 \cdot 7 = 4536$, $|B| = 5^4 = 625$, and $|A \cap B| = 5 \cdot 4 \cdot 3 \cdot 2 = 120$. Therefore

$$|A \cup B| = |A| + |B| - |A \cap B| = 4536 + 625 - 120 = \mathbf{5041}.$$

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(3.7) 4c

How many 6-letter lists (from the alphabet $\{\text{T, H, E, O, R, Y}\}$, with repetition allowed) are there in which the sequence of letters (T, H, E) appears consecutively (in that order)?

Solution:

The lists (or equivalently strings) to be counted partition into 4 sets:

$$A = \{(\text{T, H, E}, -, -, -)\},$$
$$B = \{(-, \text{T, H, E}, -, -)\},$$
$$C = \{(-, -, \text{T, H, E}, -)\}, \text{ and}$$
$$D = \{(-, -, -, \text{T, H, E})\},$$

where in the symbol "−" stands for any letter in the set $\{\text{T, H, E, O, R, Y}\}$. The 4-set version of the Inclusion-Exclusion principle says

$$\begin{aligned} |A \cup B \cup C \cup D| = \; & |A| + |B| + |C| + |D| \\ & -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & -|A \cap B \cap C \cap D| \end{aligned}$$

Notice however, because of conflicts in the list positions 2, 3, 4 and 5, that all but one of the intersections is empty. The only non-empty intersection is $A \cap D = \{\, (\text{T, H, E, T, H, E}) \,\}$, containing only one list. Observer also $|A| = |B| = |C| = |D| = 6^3 = 216$. It follows that

$$|A \cup B \cup C \cup D| = \mathbf{4} \cdot \mathbf{216} - \mathbf{1} = \mathbf{863}.$$

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<u>(3.8) 6</u>

A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, one white ball, and one black ball. You reach in and grab 20 balls. How many different outcomes are possible?

Solution:

The selections fall into 4 mutually exclusive classes.

I. 20 balls from among red, blue or green, with no white balls, and no black balls:

$$\binom{20+3-1}{20} = \binom{22}{20} = 231$$

II. 19 balls from among red, blue or green, with 1 white ball, and no black balls:

$$\binom{19+3-1}{19} = \binom{21}{19} = 210$$

III. 19 balls from among red, blue or green, with no white balls, and 1 black ball:

$$\binom{19+3-1}{19} = \binom{21}{19} = 210$$

IV. 18 balls from among red blue or green, with 1 white ball, and 1 black ball:

$$\binom{18+3-1}{18} = \binom{20}{18} = 190$$

Since each selection falls into exactly one of the above 4 classes, the addition principle provides that the total number of such selections is $\mathbf{231 + 210 + 210 + 190 = 841}$. ■

<u>(3.8) 12</u>

How many integer solutions does the equation $w + x + y + z = 100$ have if $w \geq 7$, $x \geq 0$, $y \geq 5$ and $z \geq 4$?

Solution:

Let $a = w - 7$, $b = x$, $c = y - 5$ and $d = z - 4$. The above equation becomes

$$(w-7) + x + (y-5) + (z-4) = 100 - 7 - 5 - 4$$

which is

$$a + b + c + d = 84$$

subject to the constraints $a \geq 0$, $b \geq 0$, $c \geq 0$ and $d \geq 0$. Furthermore, the two equations (together with constraints) are in one-to-one correspondance via the transformation $w = a + 7$, $x = b$, $y = c + 5$ and $z = d + 4$. Therefore the number of solutions to both systems (equation+constraints) is

$$\binom{84+4-1}{84} = \binom{\mathbf{87}}{\mathbf{84}} = \mathbf{105,995}.$$

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