CSE 16 8-5-20

Ex. (again)

$$\forall n \geq 1 : \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}} \leftarrow P(n)$$

Proof -

I. $P(1)$ is true ... same as before.

IIb. $\forall n > 1 : P(n-1) \Rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume

$$\sum_{k=1}^{n-1} k = \frac{(n-1)(n)}{2} \qquad \text{(incl. hyp.)}$$

must show

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{(incl. concl.)}$$

So

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n$$

$$= \left(\frac{n(n-1)}{2}\right) + n \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \frac{n^2 - n + 2n}{2}$$

$$= \frac{n^2 + n}{2}$$

$$= \frac{n(n+1)}{2} .$$

Result follows by P.M.I. ∎

Exercise

re-do all previous examples in form II b.

Another variation is strong induction, or 2nd P.M.I. By contrast IIa, IIb are called weak induction or 1st P.M.I.

strong Induction

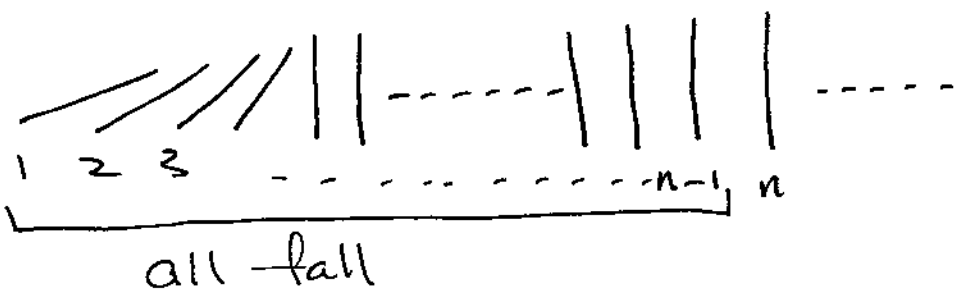
I. show $P(1)$

IIc. $\forall n \geq 1 : \underbrace{(P(1) \wedge P(2) \wedge \cdots \wedge P(n))}_{\text{ind. hypothesis}} \Rightarrow \underbrace{P(n+1)}_{\text{ind. concl.}}$

or

IId. $\forall n > 1 : \underbrace{(P(1) \wedge P(2) \wedge \cdots \wedge P(n-1))}_{\text{ind. hyp.}} \Rightarrow \underbrace{P(n)}_{\text{ind. concl.}}$

Domino analogy:



Theorem

Every positive integer can be expressed as the product of (zero or more) primes.

Rmk.
This is half of F.T.A., i.e. existence of prime factorization.

Proof.
We show

$$\forall n \geq 1 : \boxed{n \text{ is a product of primes}}$$

($P(n)$)

I. $P(1)$ is true since 1 is the empty product of primes.

IId. $\forall n > 1 : (P(1) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume for all k in range $1 \leq k < n$ that k can be expressed as a product of Primes.

we must show that n is a product of Primes.

<u>Case 1</u> : n is Prime

In this case, n is a Product of one Prime, itself

<u>case 2</u> n is composite.

In this case $n = ab$ for some $a, b \in \mathbb{Z}$ with $1 < a < n$, $1 < b < n$.

By the induction hypothesis both a and b are Products of Primes.

i.e.

$$a = p_1 p_2 \cdots p_k$$

$$b = q_1 q_2 \cdots q_\ell$$

where p_i, q_j are primes $(1 \le i \le k,\ 1 \le i \le \ell)$.

Thus

$$n = ab = (p_1 \cdots p_k)(q_1 \cdots q_\ell)$$

is also a product of primes.

Result follows for all $n \ge 1$ by the 2nd PMI. ∎

<u>Lemma 1</u> (ch. 7 #30)

Let $a, b, p \in \mathbb{Z}$. Suppose p is prime.

If $p \mid ab$ then $p \mid a$ or $p \mid b$

Lemma 2

Let $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ and let p be a prime. If $p \mid (a_1 a_2 \cdots a_k)$, then $p \mid a_i$ for some $1 \le i \le k$.

Exercise Prove this.

Theorem

The prime factorization of every $n \ge 1$ is unique, i.e. if

$$p_1 \cdot p_2 \cdots p_k = n = q_1 q_2 \cdots q_l$$

where $p_1 \le p_2 \le \cdots \le p_k$, $q_1 \le q_2 \le \cdots \le q_l$ are prime numbers, then $k = l$ and $p_i = q_i$ $(1 \le i \le k)$.

Rmk: This is uniqueness in F.T.A.

Proof

Let $P(n)$ be the statement: "the prime-factorization of n is unique".

I. $P(1)$ is true since there is only one empty product.

II. $\forall n > 1 : (P(1) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$.

Let $n > 1$ be arbitrary. Assume the prime-factorization of any positive integer less than n is unique. We must show the factorization of n into primes is unique.

Suppose

* $p_1 p_2 \cdots p_k = n = q_1 q_2 \cdots q_l$

are 2 factorizations of n into primes, with $p_1 \le \cdots \le p_k$ and $q_1 \le \cdots \le q_l$.

Then $n = (p_1 p_2 \cdots p_{k-1}) p_k$, so $p_k \mid n$.

Hence $p_k \mid q_1 q_2 \cdots q_l$. By lemma 2, $p_k \mid q_j$ for some $1 \le j \le l$. By re-indexing we have $p_k \mid q_l$. But q_l has only $\{1, q_l\}$ as divisors. Since $p_k \neq 1$, we have $p_k = q_l$.

Cancel $p_k = q_l$ on both sides of (*), and define m by

$$p_1 p_2 \cdots p_{k-1} = m = q_1 q_2 \cdots q_{l-1}.$$

since $p_k > 1$ we have $m = \frac{n}{p_k} < n$.

By the induction hypothesis, the factorization of m into primes is unique, i.e. $k-1 = l-1$ and

$$p_i = q_i \quad \text{for } 1 \le i \le k-1.$$

since $p_k = q_l$ we have $k = l$ and

$$p_i = q_i \quad \text{for } 1 \le i \le k.$$

Thus the prime factorization of n is unique.

Result follows for all $n \ge 1$ by 2nd P.M.I.

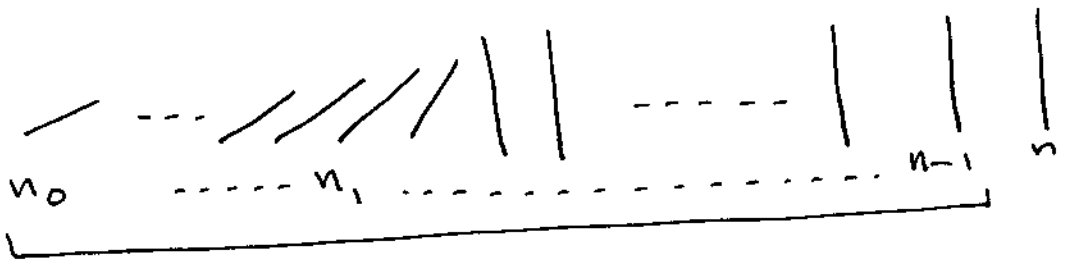
strong induction with multiple base cases

To show $\forall n \geq n_0 : P(n)$ prove

I. $P(n_0), P(n_0+1), \ldots, P(n_1)$

(n_0: smallest base case; n_1: largest base case)

II. $\forall n > n_1 : \big(P(n_0) \wedge \cdots \wedge P(n-1)\big) \Rightarrow P(n)$.



$P(n)$

Ex. $\forall n \geq 8$ $\boxed{\exists\, x, y \geq 0 \text{ s.t. } n = 3x + 5y}$

note. $P(1), \ldots, P(7)$ are all false, since at least one of x or y would have to be negative.

Proof.

I $P(8)$ is true since $8 = 3\cdot 1 + 5\cdot 1$

$P(9)$ " " " $9 = 3\cdot 3 + 5\cdot 0$

$P(10)$ " " " $10 = 3\cdot 0 + 5\cdot 2$

II d. $\forall n > 10 : (P(8) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$

Let $n > 10$ be arbitrary. Assume for all k in range $8 \leq k < n$ that

$$\exists\, x_k, y_k \overset{\geq 0}{} \text{ s.t. } k = 3x_k + 5y_k$$

we must show

$$\exists\, x, y \geq 0 : n = 3x + 5y$$

⋮

continue