cse 16 8-4-20

Exercise $\forall n \geq 1$: $\boxed{\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2}$ $\leftarrow P(n)$

Proof.

I. $P(1)$ says $\sum_{k=1}^{1} k^3 = \left(\frac{1\cdot(1+1)}{2}\right)^2$, i.e.

$1^3 = 1^2$, i.e. $1 = 1$ ✓.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$ be chosen arbitrarily. Assume

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \text{(ind. hyp.)}$$

we must show

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \quad \text{(ind. concl.)}$$

$\to 0$

$$\sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n} k^3\right) + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \left\{\begin{array}{l}\text{by the}\\ \text{ind. hyp.}\end{array}\right.$$

$$= \frac{n^2(n+1)^2 + 4\cdot(n+1)^3}{4}$$

$$= (n+1)^2 \cdot \frac{n^2+4(n+1)}{2^2}$$

$$= \frac{(n+1)^2(n^2+4n+4)}{2^2}$$

$$= \frac{(n+1)^2(n+2)^2}{2^2}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

Result follows for all $n \geq 1$ by the P.M.I. $\blacksquare$

Rmk

$P(n)$ may not be true for some finite # of initial terms.

$$\underbrace{P(1)\ P(2)\ \cdots P(n_0-1)}_{\text{false}}\ \underbrace{P(n_0)\ P(n_0+1)\ \cdots\cdots}_{\text{true}}$$

To Prove

$$\forall n \geq n_0 : P(n)$$

we show

I. $P(n_0)$ is true

II. $\forall n \geq n_0 : P(n) \Rightarrow P(n+1)$

Ex. $\forall n \geq 4$: $\boxed{5n+8 < n^2+4n+1}$ ⇐ $P(n)$

check: $P(1), P(2), P(3)$ are false.

Proof.

I. $P(4)$ is $5\cdot 4+8 < 4^2+4\cdot 4+1$,

i.e. $28 < 33$, which is true.

II. $\forall n \geq 4 : P(n) \Rightarrow P(n+1)$

Let $n \geq 4$ be arbitrary. Assume

$$5n+8 < n^2+4n+1 . \quad \text{(ind. hyp.)}$$

we must show

$$5(n+1)+8 < (n+1)^2+4(n+1)+1 . \quad \text{(ind. concl.)}$$

so

$$5(n+1)+8 = (5n+8)+5$$

$$< (n^2+4n+1)+5 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= n^2+4n+6$$

$$< n^2+4n+6+2n \quad \left\{ \begin{array}{l} \text{since } n \geq 4 \\ \Rightarrow 2n \geq 8 > 0 \end{array} \right.$$

$$= n^2+6n+6$$

$$= (n+1)^2+4(n+1)+1 \quad \left\{ \begin{array}{l} \text{by some} \\ \text{algebra. *} \end{array} \right.$$

<u>note</u>: $(n+1)^2+4(n+1)+1$

$$= n^2+2n+1+4n+4+1$$

$$= n^2+6n+6 = (n^2+4n+6)+2n$$

Result follows for all $n \geq 1$ by

P.M.I.

Ex. (ch. 10 #17)

Let $A_1, A_2, A_3, \ldots$ be an infinite seq. of subsets of $\mathcal{U}$.

$$\forall n \geq 1 : \boxed{\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k}} \leftarrow P(n)$$

note:

$n=1$: $\overline{A_1} = \overline{A_1}$

$n=2$: $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$ (DeMorgan)

$n=3$: $\overline{A_1 \cap A_2 \cap A_3} = \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$

$\vdots$

$n=$ $\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$

Rmk: This is DeMorgan's law extended to any finite # of sets.

<u>Proof.</u>

I. $P(1)$ says $\overline{A_1} = \overline{A_1}$ which is true.

II. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$.

Let $n \geq 1$ be arbitrary. Assume

$$\overline{\bigcap_{k=1}^{n} A_k} = \bigcup_{k=1}^{n} \overline{A_k} \qquad \text{(ind. hyp.)}$$

we must show

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \bigcup_{k=1}^{n+1} \overline{A_k} \qquad \text{(ind. concl.)}$$

so

$$\overline{\bigcap_{k=1}^{n+1} A_k} = \overline{\left(\bigcap_{k=1}^{n} A_k\right) \cap A_{n+1}}$$

$$= \left(\overline{\bigcap_{k=1}^{n} A_k}\right) \cup \overline{A_{n+1}} \quad \Big\{ \text{by DeMorgan's law}$$

$$= \left(\bigcup_{k=1}^{n} \overline{A_k}\right) \cup \overline{A_{n+1}} \quad \Big\{ \text{by the ind. hyp.}$$

$$= \bigcup_{k=1}^{n+1} \overline{A_k} \, .$$

Result follows by P.M.I. $\blacksquare$

<u>Exercise</u>: (ch. 10 #18)

$$\forall n \geq 1 : \overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

<u>Exercise</u>.

Let $P_1, P_2, P_3, \dots$ be an infinite seq. of statements.

<u>notation</u>

- $\bigvee_{k=1}^{n} P_k = P_1 \vee P_2 \vee P_3 \vee \dots \vee P_n$
- $\bigwedge_{k=1}^{n} P_k = P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n$

show that

- $\forall n \geq 1 : \sim\left(\bigwedge_{k=1}^{n} P_k\right) = \bigvee_{k=1}^{n} (\sim P_k)$
- $\forall n \geq 1 : \sim\left(\bigvee_{k=1}^{n} P_k\right) = \bigwedge_{k=1}^{n} (\sim P_k)$

Proof of P.M.I.

Recall

well ordering Property of $\mathbb{N}$

Any non-empty subset of $\mathbb{N}$ contains a least element.

Theorem (PMI)

Given any Prop. fcn. $P(n)$ on domain $\mathbb{N}$, the statement

$$\left[P(1) \wedge \forall n\left(P(n) \Rightarrow P(n+1)\right)\right] \Rightarrow \forall n\, P(n)$$

is true.

Proof.

Assume both $P(1)$ and $\forall n\left(P(n) \Rightarrow P(n+1)\right)$ are true. We must show $\forall n\, P(n)$ is also true.

Let $S \subseteq \mathbb{N}$ be defined as

$$S = \{n \in \mathbb{N} \mid \sim P(n)\},$$

i.e. S is the set of pos. ints for which $P(n)$ is false.

It will be suff. to show $S = \emptyset$, for then $P(n)$ is true for all $n \in \mathbb{N}$, as required.

Assume, to get a $\times$, that $S \neq \emptyset$. Then by the W.O.P., S contains a least element, call it m. So $m \in S$ and $m-1 \notin S$.

Since $P(1)$ is true, we have $1 \notin S$, so $m \neq 1$, hence $m \geq 2$, and $m-1 \geq 1$. Thus $m-1 \in \mathbb{N}$.

Since $\forall n\,(P(n) \Rightarrow P(n+1))$ is true, we have, by letting $n = m-1$, that

$$P(m-1) \Rightarrow P(m)$$

is true. But recall $m-1 \notin S$, so

$$P(m-1)$$

is also true. It follows that

$$P(m)$$

is true (by Modus Ponens.). But this implies $m \notin S$, contradicting that $m \in S$.

This $\times$ shows our assumption was false, hence $S = \emptyset$, as required.

$\blacksquare$

Variations on Induction

I. $P(1)$

IIa. $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

IIb. $\forall n > 1 : P(n-1) \Rightarrow P(n)$

Rmk.
this is just change of variable, replace n in IIa by $(n-1)$. (note $n-1 \geq 1 \Rightarrow n \geq 2 \Rightarrow n > 1$.)

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (again).

Proof.
I. $P(1)$ ... same.

IIb. ... exercise ...