CSE 16 8-12-20

11.2 Properties of Relations

Defn

we call $R \subseteq A \times A$

- reflexive iff $\forall x \in A$

$$x R x$$

- symmetric iff $\forall x, y \in A$

$$x R y \Rightarrow y R x$$

- transitive iff $\forall x, y, z \in A$

$$x R y \wedge y R z \Rightarrow x R z$$

- antisymmetric iff $\forall x, y \in A$

$$x R y \wedge y R x \Rightarrow x = y$$

Recall standard relations on $\mathbb{Z}$

Relation	Ref.	symm.	trans.	antisymm.
$<$	no	no	yes	yes
$\leq$	yes	no	yes	yes
$>$	no	no	yes	yes
$\geq$	yes	no	yes	yes
* $=$	yes	yes	yes	yes
$\neq$	no	yes	no	no
$\mid$	yes	no	yes	yes
* $\equiv_m$	yes	yes	yes	no

Exercise

Prove all 32 entries in this table.

- $\equiv_m$ is transitive

must show $x \equiv_m y \wedge y \equiv_m z \Rightarrow x \equiv_m z$ ✓

$$\left.\begin{array}{l} x \equiv_m y \Rightarrow m \mid (x-y) \\ y \equiv_m z \Rightarrow m \mid (y-z) \end{array}\right\} \Rightarrow m \mid (x-\cancel{y})+(\cancel{y}-z)$$

$$\Rightarrow m \mid (x-z)$$

$$\Rightarrow x \equiv_m z \; . \quad \blacksquare$$

- $\mid$ is <u>not</u> symmetric.

must show $\forall x,y: x \mid y \Rightarrow y \mid x$ is false

i.e. show $\sim \forall x,y: x \mid y \Rightarrow y \mid x$ is true

i.e $\exists\, x,y: x \mid y \wedge y \nmid x$

Let $x=2$, $y=4$. Then $2 \mid 4$ but $4 \nmid 2$.

$\blacksquare$

- $<$ is antisymmetric

must show $\forall x,y: x<y \wedge y<x \Rightarrow x=y$

notice the hypothesis $x<y \wedge y<x$ is a contradiction, never true for any x,y. $\therefore$ implication is true (vacously).

Digraph interpretation

- reflexive: $xRx \quad (\forall x)$

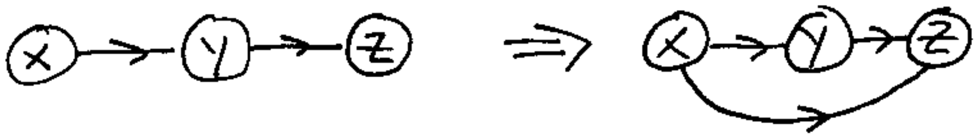
every vertex must have a loop

- <u>symmetric</u> : $xRy \Rightarrow yRx \quad (\forall x, y)$

equivalently :

not Possible   Possible

- <u>transitive</u> $xRy \wedge yRz \Rightarrow xRz \quad (\forall x, y, z)$

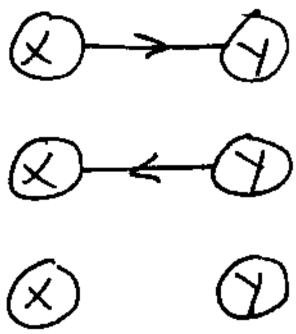


if $x = z$, this becomes

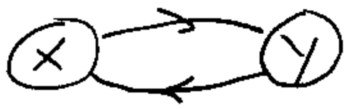
- <u>antisymmetric</u>: $xRy \wedge yRx \Rightarrow x = y \ (\forall x, y)$

equivalently: given $x \neq y$

Possible



not Possible



Problems 11.2 (P. 208-209) # 2, 4, 6

11.3 Equivalence Relations

Defn

A relation $R \subseteq A \times A$ on A is called an equivalence relation iff it is

(i) reflexive

(ii) symmetric

(iii) transitive

The table from 11.2, has only 2 equivalence relations

- $=$ on $\mathbb{Z}$: $\{(x,x) \mid x \in \mathbb{Z}\}$
- $\equiv_m$ on $\mathbb{Z}$: $\{(x,y) \mid m \mid (x-y),\ x,y \in \mathbb{Z}\}$

Lets complete proof that $\equiv_m$ is an eq. relation.

- $\equiv_m$ is reflexive: $x \equiv_m x \ (\forall x)$

$$m \mid 0 \Rightarrow m \mid (x-x) \Rightarrow x \equiv x \pmod{m}$$

- $\equiv_m$ is symmetric: $x \equiv_m y \Rightarrow y \equiv_m x \ (\forall x, y)$

$$\begin{aligned} x \equiv y \pmod{m} &\Rightarrow m \mid (x-y) \\ &\Rightarrow m \mid -(x-y) \\ &\Rightarrow m \mid (y-x) \\ &\Rightarrow y \equiv x \pmod{m} \end{aligned}$$

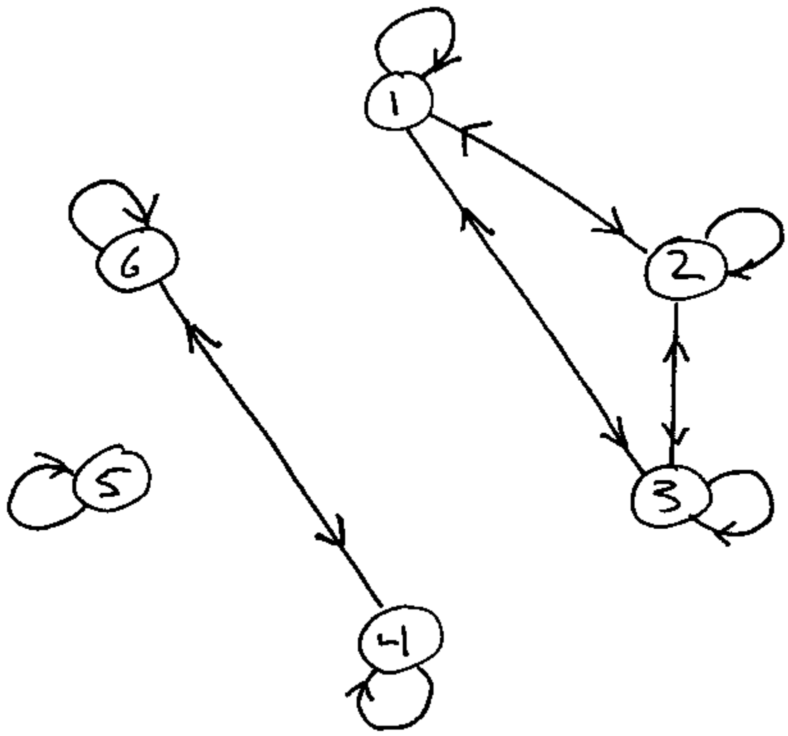
<u>note</u>:
$m \mid a \Rightarrow m \mid (-a)$ since $a = km \Rightarrow (-a) = (-k)m$

- $\equiv_m$ is transitive (prev. proved.)

Ex $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3),$
$(3,1), (3,2), (3,3), (4,4), (4,6),$
$(5,5), (6,4), (6,6)\}$

Digraph.



one checks that i, ii & iii of defn are satisfied.

Defn

Let $R \subseteq A \times A$ be an equivalence relation, and let $a \in A$. The <u>equivalence class containing a</u>, denoted by $[a]$ is the set

$$[a] = \{ x \in A \mid x R a \} \subseteq A .$$

i.e. $[a]$ is the set of all elements equivalent to a.

Ex Previous ex. on $A = \{1, 2, 3, 4, 5, 6\}$

$$[1] = [2] = [3] = \{1, 2, 3\}$$

$$[4] = [6] = \{4, 6\}$$

$$[5] = \{5\}$$

Lemma

Let R be an eq. relation on A:
($R \subseteq A \times A$ satisfies i, ii & iii.)

Then for any $a, b \in A$

$$a R b \quad \overset{\Longrightarrow}{\underset{\Longleftarrow}{\text{iff}}} \quad [a] = [b]$$

Proof.

($\Rightarrow$) Suppose aRb. It's sufficient to show: $x \in [a] \Leftrightarrow x \in [b]$, for then $[a] = [b]$.

$x \in [a]$	$\Rightarrow x R a$	def.
	$\Rightarrow x R a \wedge a R b$	hyp.
	$\Rightarrow x R b$	trans.
	$\Rightarrow x \in [b]$	def.

$$
\begin{aligned}
x \in [b] &\Rightarrow x R b && \text{def.} \\
&\Rightarrow x R b \wedge a R b && \text{hyp.} \\
&\Rightarrow x R b \wedge b R a && \text{symm.} \\
&\Rightarrow x R a && \text{trans} \\
&\Rightarrow x \in [a] && \text{def.}
\end{aligned}
$$

Therefore $[a]=[b]$, as claimed.

($\Leftarrow$) Suppose $[a]=[b]$. Then aRa by reflexivity of R.

$$
\begin{aligned}
&\therefore\ a \in [a] && \text{def.} \\
&\therefore\ a \in [b] && \text{hyp.} \\
&\therefore\ a R b && \text{def.}
\end{aligned}
$$

$\blacksquare$

<u>Rmk</u>
notice we used all 3 properties ref., symm. & trans.

for $m \in \mathbb{N}$, what are the eq. classes of $\equiv_m$ on $\mathbb{Z}$.

By last lemma

$$
\begin{aligned}
x \in [a] \quad &\text{iff} \quad x \equiv_m a \\
&\text{iff} \quad m \mid (x-a) \\
&\text{iff} \quad x - a = km \quad (\text{some } k \in \mathbb{Z}) \\
&\text{iff} \quad x = km + a \qquad \text{" \quad "}
\end{aligned}
$$

Ex. $m = 2$

$x \in [0]$ iff $x = 2k$ iff x is even

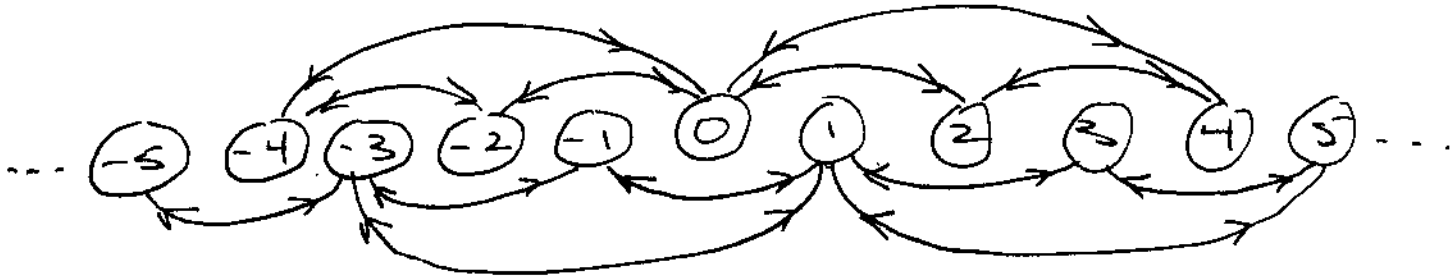
$x \in [1]$ iff $x = 2k+1$ iff x is odd.

so

$[0] = [2] = [4] = \cdots = \{\text{even numbers}\}$

$[1] = [3] = [5] = \cdots = \{\text{odd numbers}\}$

Digraph



Ex. $m = 3$

$x \in [0]$ iff $x = 3k$ (some $k \in \mathbb{Z}$)

$x \in [1]$ iff $x = 3k+1$

$x \in [2]$ iff $x = 3k+2$

so

$[0] = [3] = [6] = \dots = \{3k \mid k \in \mathbb{Z}\}$

$[1] = [4] = [7] = \dots = \{3k+1 \mid k \in \mathbb{Z}\}$

$[2] = [5] = [8] = \dots = \{3k+2 \mid k \in \mathbb{Z}\}$

for any $m \in \mathbb{Z}$, $x \in \mathbb{Z}$

$$x = km + r \quad (0 \le r < m)$$

by div. alg. This is eq. to

$$x \equiv_m r$$

This is eq. to

$$x \in [r]$$

Thus

$$[r] = [m+r] = [2m+r] = \dots = \{km+r \mid k \in \mathbb{Z}\}.$$

The eq. classes

$$[0], [1], [2], \dots, [m-1]$$

are <u>all</u> of the eq. classes of $\equiv_m$

<u>Probleme</u> 11.3 (p. 214) # 2, 4, 6