CSE 16 8-11-20

closed formula for n^{th} Fibonacci number.

Define

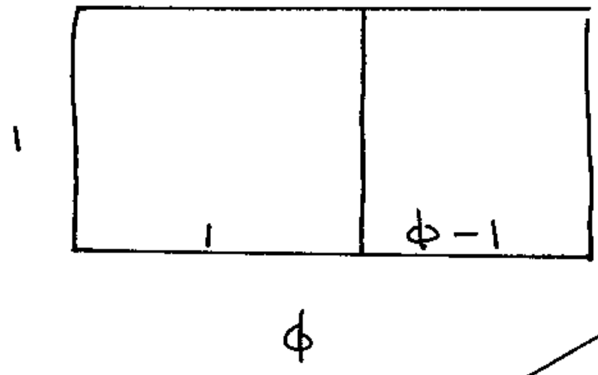
$$\phi_1 = \frac{1+\sqrt{5}}{2} \leftarrow \text{Golden ratio}$$

$$\phi_2 = \frac{1-\sqrt{5}}{2} \longleftarrow \text{conjugate of G.R.}$$

<u>note</u>: these are roots of

$$* \qquad x^2 - x - 1 = 0$$

what's special about this eqn, and whats 'golden' about G.R. ?



$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

$$\phi(\phi - 1) = 1$$

$$\phi^2 - \phi = 1$$

$$\phi^2 - \phi - 1 = 0$$ which is *

or $$\boxed{\phi^2 = \phi + 1}$$

<u>Theorem</u>

$$\forall n \geq 1: \boxed{F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right)}$$ $P(n)$

use strong ind. IId, with 2 base cases

I. • if $n=1$

$$RHS = \frac{1}{\sqrt{5}}\left(\phi_1^1 - \phi_2^1\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{\cancel{1}+\sqrt{5} - \cancel{1}+\sqrt{5}}{\sqrt{5}\cdot 2} = \frac{2\sqrt{5}}{2\sqrt{5}}$$

$$= 1 = F_1 = LHS$$

• if $n=2$

$$RHS = \frac{1}{\sqrt{5}}\left(\phi_1^2 - \phi_2^2\right)$$

$$= \frac{1}{\sqrt{5}}\left((\phi_1 + \cancel{1}) - (\phi_2 + \cancel{1})\right)$$

$$= 1 \quad \text{(by above calculation)}$$

$$= F_2 = LHS$$

II d. $\forall n > 2 : (P(1) \wedge \dots \wedge P(n-1)) \Rightarrow P(n)$

Let $n > 2$ be arbitrary. Assume for all k in the range $1 \le k < n$ that

$$F_k = \frac{1}{\sqrt{5}}\left(\phi_1^k - \phi_2^k\right), \quad \text{ind. hyp.}$$

we must show that

$$F_n = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right) \quad \text{ind. concl.}$$

so

$$RHS = \frac{1}{\sqrt{5}}\left(\phi_1^n - \phi_2^n\right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi_1^2 \cdot \phi_1^{n-2} - \phi_2^2 \cdot \phi_2^{n-2}\right)$$

$$= \frac{1}{\sqrt{5}}\left((\phi_1 + 1)\phi_1^{n-2} - (\phi_2 + 1)\phi_2^{n-2}\right)$$

$$= \frac{1}{\sqrt{5}} \left(\left(\phi_1^{n-1} + \phi_1^{n-2}\right) - \left(\phi_2^{n-1} + \phi_2^{n-2}\right) \right)$$

$$= \frac{1}{\sqrt{5}} \left(\left(\phi_1^{n-1} - \phi_2^{n-1}\right) + \left(\phi_1^{n-2} - \phi_2^{n-2}\right) \right)$$

$$= \frac{1}{\sqrt{5}} \left(\phi_1^{n-1} - \phi_2^{n-1}\right) + \frac{1}{\sqrt{5}} \left(\phi_1^{n-2} - \phi_2^{n-2}\right)$$

$= F_{n-1} + F_{n-2}$ { by the ind. hyp. with $k = n-1$, $k = n-2$

$= F_n$ { by the Fib. recurrence.

Result follows for all n by 2^{nd} PMI. ∎

Exercise: try to do this by weak ind.

more exercises:

- ch. 10 #25 : $\sum_{k=1}^{n} F_k = F_{n+2} - 1$.
- ch. 10 #29 : $\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$.

hint: $\binom{a}{b} = 0$ when $b < 0$ or $b > a$.

chapter 11 : Relations

(11.1) general relations

Defn

let A, B be sets. A relation from A to B is a subset

$$R \subseteq A \times B$$

domain codomain

notation given $x \in A$, $y \in B$ write

- xRy to mean $(x,y) \in R$
- $x \not R y$ " " $(x,y) \notin R$

more generally: given sets $A_1, A_2, \ldots, A_n$

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n$$

used in data base theory

Defn

A relation on A is a relation from A to A, i.e.

$$R \subseteq A \times A$$

Examples of relations on $\mathbb{Z}$.

- $<$ is the set $\{(x,y) \mid x<y\} \subseteq \mathbb{Z}\times\mathbb{Z}$
- $\leq$ " " " $\{(x,y) \mid x\leq y\} \subseteq \mathbb{Z}\times\mathbb{Z}$
- $>$ " " " ... $x>y$ .....
- $\geq$ " " " .... $x\geq y$ .....
- $=$ .... $\{(x,x) \mid x\in\mathbb{Z}\}$
- $\neq$ .... $\mathbb{Z}\times\mathbb{Z} - \{(x,x) \mid x\in\mathbb{Z}\}$

- $|$ is the set

$$\{(x,y) \mid \exists k : y = kx\}$$

- for any $m \in \mathbb{N}$, $\equiv_m$ is the set $\{(x,y) \mid m \mid (x-y)\}$

Relations on finite sets

- let $A = \{1,2,3,4\}$. let

$$R = \{(1,1),(2,1),(2,2),(3,3),(3,2),(3,1),(4,4),(4,3),(4,2),(4,1)\}$$

Thus $4R1$ and $3R2$ but $3 \not{R} 4$.

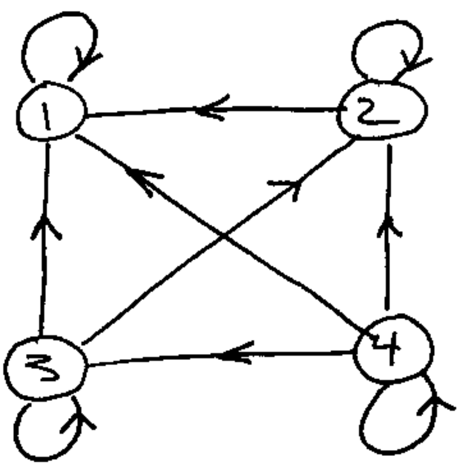
<u>note</u>: $|A \times A| = |A| \cdot |A| = 4 \cdot 4 = 16$

(so # of relations on A) $= 2^{16}$

Relations on a finite set can be dipicted as a <u>directed graph</u> (<u>digraph</u>)

$$x R y \approx \quad (x) \longrightarrow (y)$$

from example on $A = \{1, 2, 3, 4\}$



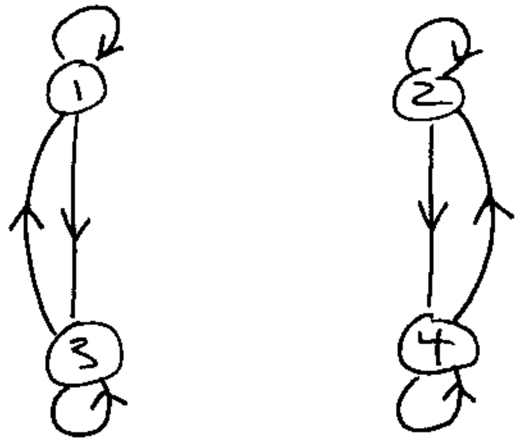
<u>note</u>: this is the $\geq$ on $A = \{1, 2, 3, 4\}$

i.e $x R y$ iff $x \geq y$.

- Again $A = \{1, 2, 3, 4\}$. define

$$S = \{(1,1), (1,3), (3,1), (3,3), (2,2), (2,4), (4,2), (4,4)\}$$

digraph representation:



so:

$2R4$

$2R2$

$4 \not R 3$

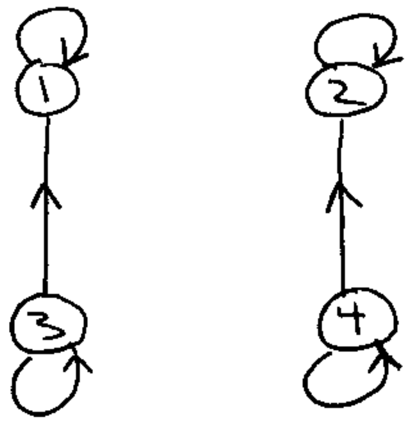
$3 \not R 4$

this is the "same parity" relation, i.e. xSy iff x and y have same parity. Equivalently $S = (\equiv_2)$

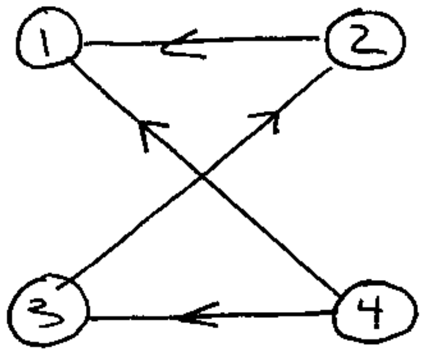
i.e. xSy iff $x \equiv y \pmod 2$.

- Since relations are sets, we can apply set operations to them.

$R \cap S$:



$R - S$:



<u>Exercise</u> Draw digraphs for

$A \times A$ $R \cup S$

$(A \times A) - S$ $R \oplus S$

$(A \times A) - R$ $S - R$

Problems (11.1) p. 204

2, 4, 6, 8, 10.

(11.2) Properties of relations

Defn

we call a relation $R \subseteq A \times A$

- reflexive iff $\forall x \in A$

$$x R x$$

---- more to come -----