CSE 16 7-9-20

Recall:

$$\binom{n}{k} = C(n,k)$$

denotes # of k-element subsets of an n-element set.

<u>Theorem</u>

If $0 \le k \le n$ then

$$\boxed{\binom{n}{k} = C(n,k) = \frac{n!}{k!\,(n-k)!}}$$

<u>Ex.</u> $$\binom{4}{3} = \frac{4!}{3!\,(4-3)!} = \frac{4!}{3!\cdot 1!} = \frac{4\cdot\cancel{3}\cdot\cancel{2}}{\cancel{3}\cdot\cancel{2}} = 4$$

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4\cdot 3\cdot\cancel{2\cdot 1}}{2\cdot 1\cdot\cancel{2\cdot 1}} = 6$$

To see why, we compute $P(n,k)$ again in a different way

To construct a k-permutation of an n-set S, perform 2 steps

1st: choose k-subset of S.

$\#\text{ways} = C(n,k)$

2nd: choose a permutation of these k elements

$\#\text{ways} = k!$

By mult. principle, we can do both steps in succession in $C(n,k)\cdot k!$ ways.

$$\therefore \quad P(n,k) = C(n,k)\cdot k!$$

Recall $P(n,k) = \dfrac{n!}{(n-k)!}$

$$\therefore \quad \frac{n!}{(n-k)!} = C(n,k)\cdot k!$$

$$\therefore \quad C(n,k) = \frac{n!}{k!\,(n-k)!}$$

<u>Ex.</u>
how many bit strings of length 10 contain exactly 6 1's.

e.g.

$$\underset{1}{0}\ \underset{2}{1}\ \underset{3}{1}\ \underset{4}{0}\ \underset{5}{1}\ \underset{6}{0}\ \underset{7}{1}\ \underset{8}{1}\ \underset{9}{0}\ \underset{10}{1}$$

corresponds to : $\{2,3,5,7,8,10\} \subseteq \{1,2,3,\dots,10\}$

To construct such a bit string choose which 6 positions $\{1,2,3,4,5,6,7,8,9,10\}$ are to be occupied by 1's. (the rest are 0's.)

so

(# such bit strings)

$= (\#\ 6\text{-subsets of a } 10\text{-set})$

$= \binom{10}{6}$

$= \dfrac{10!}{6!\,4!} = \dfrac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}$

$= 10\cdot 21 = \boxed{210}$

<u>note</u>: we could also count # ways of choosing which 4 positions are to be occupied by 0's. (The rest are 1's.)

(# such strings) $= (\#\ 4\text{-subsets of a } 10\text{-set})$

$= \binom{10}{4}$

$= \dfrac{10!}{4!\,6!} = \boxed{210}$

Theorem

If $0 \le k \le n$ then

$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic Proof

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-k)! \cdot (n-(n-k))!}$$

$$= \binom{n}{n-k} \qquad \blacksquare$$

Combinatorial Proof.

The LHS of the identity: $\binom{n}{k}$ counts the # of length n bit strings which contain exactly k 1's.

The RHS $\binom{n}{n-k}$ counts the # of length n bitstrings with exactly $(n-k)$ 0's.

So LHS & RHS count the same set of bit strings, hence are equal:

$$\binom{n}{k} = \binom{n}{n-k}$$

$\blacksquare$

(3.6) Binomial Theorem

using $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ $(0 \le k \le n)$

we compute

n	
0	$\binom{0}{0}=1$
1	$\binom{1}{0}=1,\ \binom{1}{1}=1$
2	$\binom{2}{0}=1,\ \binom{2}{1}=2,\ \binom{2}{2}=1$
3	$\binom{3}{0}=1,\ \binom{3}{1}=3,\ \binom{3}{2}=3,\ \binom{3}{3}=1$
4	$\binom{4}{0}=1,\ \binom{4}{1}=4,\ \binom{4}{2}=6,\ \binom{4}{3}=4,\ \binom{4}{4}=1$
5	$\binom{5}{0}=1,\ \binom{5}{1}=5,\ \binom{5}{2}=10,\ \binom{5}{3}=10,\ \binom{5}{4}=5,\ \binom{5}{5}=$
$\vdots$	$\vdots$

Pascal's Triangle:

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 $\boxed{15}$ 20 15 6 1
7	1 7 21 35 35 21 7 1
⋮	⋮

Properties:

(1) 1's on sides

(2) each interior entry is the sum of neighboring entries on row above

<u>Theorem</u> (Pascal's identity)

for $1 \le k \le n$

$$(*) \qquad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

<u>equivalently</u>

for $0 < k < n$

$$(**) \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

<u>Algebraic proof of (*)</u>

$$RHS = \binom{n}{k-1} + \binom{n}{k}$$

$$= \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{k \cdot n!}{k \cdot (k-1)!\,(n-k+1)!} + \frac{(n-k+1)\,n!}{k!\,(n-k+1)\cdot(n-k)!}$$

$$= \frac{k \cdot n!}{k!\,(n-k+1)!} + \frac{(n-k+1)\cdot n!}{k!\,(n-k+1)!}$$

$$= \frac{(\cancel{k} + n - \cancel{k} + 1)\cdot n!}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)\cdot n!}{k!\,((n+1)-k)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!}$$

$$= \binom{n+1}{k} = \text{LHS}. \qquad \blacksquare$$

combinatorial proof of (**)

let

$S = \{\text{len. } n \text{ bit strings with } k \text{ 1's}\}$

$A = \{x \in S \mid x \text{ ends in } 1\}$

$B = \{x \in S \mid x \text{ ends in } 0\}$

obviously $S = A \cup B$ and $A \cap B = \emptyset$, so by addition principle

$$|S| = |A| + |B|$$

- Every $x \in A$ can be obtained by taking a bit str. of len. $(n-1)$ with $(k-1)$ 1's and appending a 1. Thus

$$|A| = \binom{n-1}{k-1}$$

- every $x \in B$ can be obtained by taking a bit str. of len $(n-1)$ containing k 1's, and appending a 0. Thus

$$|B| = \binom{n-1}{k}$$

we already know $|S| = \binom{n}{k}$

$$\therefore \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \blacksquare$$

<u>Exercise</u>

- write an algebraic proof of (**)
- write a combinatorial proof of (*)