cse 16 7-8-20

(3.3) Addition & subtraction Principles

Ex.
how many strings of length 4 from alphabet {a, b, c, d, e, f} satisfy:

- no repeated letters
- must contain the leter a

4 types of such strings

I. $\underline{a}\ \underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}$ #strings $= 5\cdot4\cdot3 = 60$

II. $\underset{5}{\_}\ \underline{a}\ \underset{4}{\_}\ \underset{3}{\_}$ " $= 5\cdot4\cdot3 = 60$

III. $\underset{5}{\_}\ \underset{4}{\_}\ \underline{a}\ \underset{3}{\_}$ " $= 60$

IV. $\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}\ \underline{a}$ " $= 60$

$\boxed{240}$

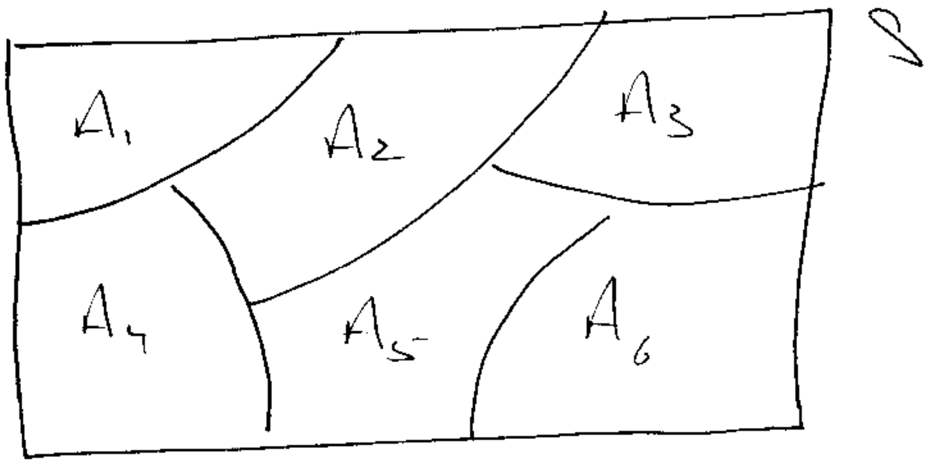
<u>Defn</u>

A <u>Partition</u> of a set S is a collection of subsets $A_1, A_2, \dots, A_n$ (non-empty) of S, such that

(1) pairwise disjoint:

$$A_i \cap A_j = \phi \quad \text{for} \quad 1 \le i < j \le n$$

(2) $\displaystyle\bigcup_{i=1}^{n} A_i = S$



Addition Principle

if $A_1, A_2, \ldots, A_n$ is a Partition of a finite set S, then

$$|S| = |A_1| + |A_2| + \cdots + |A_n|$$

(Actually, formula works even if some A_i is empty.)

Ex
how many 4-digit numbers are there satisfying

- number is even
- no 0's, i.e. only $\{1, 2, 3, \ldots, 9\}$ appear
- exactly one 5
- exactly one 4, appearing to left of the 5.

note a number is even iff its last (rightmost) digit is even $\{2, \cancel{4}, 6, 8\}$ (no 0's).

Partition:

I. $\frac{4}{\ }\ \frac{5}{\ }\ \frac{\ }{7}\ \frac{\ }{3}$ $\quad$ # numbers $= 7 \cdot 3 = 21$

II. $\frac{4}{\ }\ \frac{\ }{7}\ \frac{5}{\ }\ \frac{\ }{3}$ $\quad$ " $= 21$

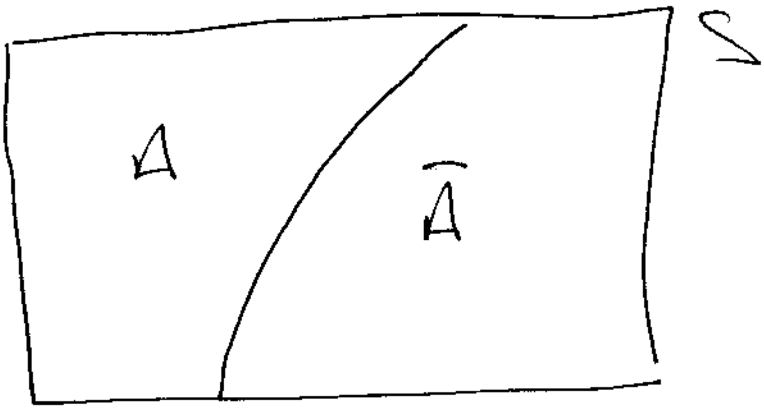
III. $\frac{\ }{7}\ \frac{4}{\ }\ \frac{5}{\ }\ \frac{\ }{3}$ $\quad$ " $= 21$

by the addition, the # of numbers is $21 + 21 + 21 = \boxed{63}$

Special case:

$A \subseteq S$ with $A \neq \emptyset$, and $A \neq S$

Partition: A, $\bar{A} = S - A$



Thus $|S| = |A| + |\bar{A}|$

$= |A| + |S - A|$

$\therefore \quad \boxed{|S - A| = |S| - |A|}$

Subtraction Principle

Ex

how many strings of length 6 from $\{a, b, c, d, e\}$ contain at least 1 d ?

let

$S = \{$strs. of len. 6 over $\{a, b, c, d, e\}\}$

$A = \{$strs. in S not containing d$\}$

$= \{$str. of len. 6 over $\{a, b, c, e\}\}$

note $A \subseteq S$ and

$S - A = \{$strs. in S containing at least one d$\}$

so by sub. Principle

$$|S-A| = |S| - |A| = 5^6 - 4^6$$

$$= 15{,}625 - 4{,}096 = \boxed{11{,}529}$$

Exercise:

re do this example using addition Principle

Types (Partition)

- (# d's) = 1
- " = 2
- " = 3
- " = 4
- " = 5
- " = 6

(3.4) Permutations

Defn

A Permutation of a finite set S is an ordered arrangement (a list or string with no repetition) of all its elements.

Ex. $S = \{1, 2, 3\}$

Permutations of S:

123
132
213
231
312
321

$\#$ permutations $= \boxed{6}$

Ex. $S = \{1, 2\}$

Permutations: 12, 21

$\#$ permutations $= \boxed{2}$

Theorem
If $|S| = n$, then the $\#$ of permutations of S is

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

(follows from mult. principle.)

Defn

A <u>k-permutation</u> of a finite set S is an ordered arrangement of k of its elements.

Ex. $S = \{1, 2, 3\}$

		#
$k=0$:	ϕ	1
$k=1$:	1, 2, 3	3
$k=2$:	12, 21, 13, 31, 23, 32	6
$k=3$:	123, 132, 213, 231, 312, 321	6

The number of k-permutations of an n-element set is denoted

$$P(n, k)$$

Observe that

- $P(n,0) = 1$
- $P(n,n) = n!$
- $P(n,k) = 0$ if $k > n$

other notation P_k^n, ${}_nP_k$, ${}^nP_k$

By the multiplication Principle

$$P(n,k) = \begin{cases} n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$

<u>note</u>:

$$\frac{n!}{(n-k)!} = \frac{n\cdot(n-1)\cdot\cdots\cdot(n-k+1)\cancel{(n-k)}\cdots\cancel{2}\cdot\cancel{1}}{\cancel{(n-k)}\cancel{(n-k-1)}\cdots\cdots\cancel{2}\cdot\cancel{1}}$$

Ex.

list all 3-permutations of $\{1, 2, 3, 4\}$

123	213	312	412
132	231	321	421
124	214	314	413
142	241	341	431
134	234	324	423
143	243	342	432

$$P(4,3) = 4 \cdot 3 \cdot 2 = \boxed{24}$$

Ex. (3.4 # 7)

find # of 9-digit numbers satisfying

- no 0's, so $\{1, 2, 3, \dots, 9\}$
- no repetition
- all odd digits are to the left of all even digits.

solution:

$O = \{\text{odd digits}\} = \{1, 3, 5, 7, 9\}$

$E = \{\text{even digits}\} = \{2, 4, 6, 8\}$

we choose

(Permutation of O)(Perm. of E)

$$\#\text{choices} = (5!)(4!)$$

$$= (120)(24)$$

$$= \boxed{2,880}$$

<u>Reminder</u>:

- see examples in section
- see solutions to odd problems.

(3.5) Combinations (counting subsets)

Defn

A *k-combination* of a finite set S is a k-element subset of S, i.e. an unordered arrangement of k of its elements.

notation

if $|S| = n$, the number of k-combinations of S is denoted

$$C(n,k) \quad \text{or} \quad \binom{n}{k}$$

read this as: "n-choose-k"

Ex. $S = \{1, 2, 3\}$

		$C(3,k)$
$k=0$:	ϕ	1
$k=1$:	$\{1\}, \{2\}, \{3\}$	3
$k=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	3
$k=3$:	$\{1, 2, 3\}$	1

Ex. list all 3-element subsets of $S = \{1,2,3,4\}$

$$\left.\begin{array}{l} \{1,2,3\} \\ \{1,2,4\} \\ \{1,3,4\} \\ \{2,3,4\} \end{array}\right] \quad C(4,3) = \binom{4}{3} = 4$$

Exercise find $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}$ and $\binom{4}{4}$

ans. $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{4} = 1$