cse 16 7-2-20

(2.7) Quantifiers

so far : Propositional logic
zeroth order logic

now: First order logic
Predicate logic
Quantifier logic

Propositional function $P(x)$ is a sentence that becomes a statement when a value for x is assigned.

<u>Ex</u> $P(x) = (x < 7)$

$P(2) = (2 < 7)$ T

$P(10) = (10 < 7)$ F

<u>Ex.</u> $P(x) =$ 'x is a sunny day'

$P(\text{today}) =$ 'today is a sunny day' T

<u>Ex.</u> $Q(x, y) = (x + y = 10)$

$Q(3, 7) = (3 + 7 = 10)$ T

$Q(3, 8) = (3 + 8 = 10)$ F

The <u>Universe of Discourse</u> or <u>Domain</u> of a Propositional function is the set of values that can be substituted for its variable(s)

<u>notation</u>: $\mathcal{U}$, or $\mathcal{U}_x$, $\mathcal{U}_y$

one way to get a statement from a Prod. fcn. is substitution

Another way is <u>Quantification</u>.

Universal Quantifier: $\forall$ (for all)

Existential Quantifier: $\exists$ (there exists)

<u>Defn</u>

- $\forall x \in \mathcal{U} : P(x)$ or
  $\forall x \, P(x)$ (if $\mathcal{U}$ is understood)

  'for all $x \in \mathcal{U}$, $P(x)$ is true'

- $\exists x \in U : P(x)$ or

  $\exists x\, P(x)$ (if U understood)

  'there exists an $x \in U$ such that $P(x)$ is true'

  'there exists atleast one $x \in U$ such that $P(x)$ is true'

  'for some $x \in U$, $P(x)$ is true'

Ex. $P(x) = (x^2 \geq 0)$, $U = \mathbb{R}$

$\forall x\, P(x) =$ 'the square of any real number is non-negative'

Ex. $Q(x) = (x < 50)$, $U = \mathbb{R}$

$\forall x\, Q(x)$ = 'every real number is less than 50'

$\exists x\, Q(x)$ = 'there exists a real number that is less than 50'

Ex. $R(x) = (x^2 < 0)$, $U = \mathbb{R}$

$\exists x\, R(x)$ = 'there exists a real number whose square is negative'

Ex $R(x) = (x^2 < 0)$, $U = \mathbb{C}$

$\exists x\, R(x)$ = 'there exists a complex number whose square is a negative real number'

Ex. $P(x,y) = (y^2 = x)$ $\quad U_x = U_y = \mathbb{R}$

$\forall x \, \exists y \, P(x,y) =$ 'for every real x there exists a real y such that $y^2 = x$'

= 'every real number has a real square root'.

Ex. $R(x,y,z) = (x \cdot y = z)$

$U_x = U_y = U_z = \mathbb{Q}$

$\forall x \, \forall y \, \exists z : R(x,y,z)$

'for any rational numbers x, y there exists a rational number z s.t. $xy = z$.'

'the product of any two rational numbers is a rational number'.

Ex. Translate:

'every real number has a real cube root'

$$\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} : y^3 = x$$

note that

$$\exists y \in \mathbb{R} \ \forall x \in \mathbb{R} : y^3 = x$$

says something very different:

'there exists a real number that is the cube root of every real number'

(2.8) Quantifiers & conditionals

Quantification operations are often implicit in mathematical writing.

Ex. $x^2 - y^2 = (x-y)(x+y)$

$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}$ (what author meant)

This happens especially with implications and almost always $\forall$.

Ex. (x is a multiple of 6) $\Longrightarrow$ (x is even)

$\forall x \in \mathbb{Z}$

Advice: explicitly state quantifiers in your mathematical writing.

(2.9) Translation Problems

consider

'all horses in California are blue'

There are many ways to translate this into symbolic logic.

Ex $U = \{\text{horses in California}\}$

$B(x) =$ 'x is blue'

Translation: $\forall x \; B(x)$

Ex. $U = \{\text{horses}\}$

$C(x) =$ 'x lives in California'

Translation: $\forall x \, (C(x) \Rightarrow B(x))$

Ex. $U = \{\text{living things}\}$

$H(x) =$ 'x is a horse'

Translation: $\forall x\ (H(x) \wedge C(x) \Rightarrow B(x))$

Note: $T \Rightarrow P = P$, since

P	$T \Rightarrow P$
0	0
1	1

so 1st translation could have been

$$\forall x\ (T \Rightarrow B(x))$$

<u>lesson</u>: adding hypotheses effectively narrows the universe

If $S \subseteq U$, then

$$\forall x \in U \, ((x \in S) \Rightarrow P(x))$$

is the same as

$$\forall x \in S : P(x)$$

<u>Ex.</u> let $L(x,y)$ = 'x loves y', where $U_x = U_y = \{\text{people}\}$. Translate

(a) 'everybody loves somebody'

$$\forall x \, \exists y \, L(x,y)$$

(b) 'somebody loves everybody'

$$\exists x \; \forall y \; L(x,y)$$

(c) 'nobody loves everybody'

$$\sim \exists x \; \forall y \; L(x,y)$$

(d) 'there is exactly one person who everone loves'

$$\exists y \left(\forall x \; L(x,y) \wedge \forall z \left(z \neq y \Rightarrow \exists w \sim L(w,z) \right) \right)$$

(e) 'there is a person who loves everyone but themself'

$$\exists x \left(\sim L(x,x) \wedge \forall y \left(y \neq x \Rightarrow L(x,y) \right) \right)$$

Ex. $U = \{\text{people}\}$

$E(x)$ = 'x is King of England'

$F(x)$ = 'x is King of France'

translate: 'no King of England is not the King of France.'

$$\sim \exists x \, (E(x) \wedge \sim F(x))$$

(2.10) Negations

Consider a small finite universe

$$U = \{1, 2\}$$

let $P(x)$ be a prop. fcn. on this universe. Then

$$\forall x\, P(x) = P(1) \wedge P(2)$$

and

$$\exists x\, P(x) = P(1) \vee P(2)$$

Thus

$$\begin{aligned} \sim \forall x\, P(x) &= \sim (P(1) \wedge P(2)) \\ &= \sim P(1) \vee \sim P(2) \\ &= \exists x \sim P(x) \end{aligned}$$

and

$$\sim \exists_x P(x) = \sim (P(1) \vee P(2))$$
$$= \sim P(1) \wedge \sim P(2)$$
$$= \forall x \sim P(x)$$

Fact: this works for <u>all</u>
universes & all prop. fcns.