cse 16 7-16-20

notation:

$$\binom{n}{p_1, p_2, p_3, \dots, p_k} = \frac{n!}{p_1! \cdot p_2! \cdot p_3! \cdots p_k!}$$

(where $p_1 + p_2 + p_2 + \dots + p_k = n$)

this symbol is called a multinomial coefficient.

Theorem

$$\binom{n}{p_1, p_2, \dots, p_k} = \binom{n}{p_1} \cdot \binom{n-p_1}{p_2} \cdot \binom{n-p_1-p_2}{p_3} \cdots \binom{n-p_1-p_2\cdots-p_{k-1}}{p_k}$$

Exercise: Prove this

- algebraic: easy
- combinatorial: also easy !

(3.9) Pigeonhole Principle

Defn

let $x \in \mathbb{R}$.

- **floor** of x

  $\lfloor x \rfloor = (\text{greatest int} \leq x)$

- **ceiling** of x

  $\lceil x \rceil = (\text{least int} \geq x)$

Alternately, $\lfloor x \rfloor$ and $\lceil x \rceil$ are the unique integers satisfying

- $$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$
- $$\lceil x \rceil - 1 < x \leq \lceil x \rceil$$

Equivalently:

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

Pigeonhole Principle

Suppose n objects are placed in k boxes. Then

(a) at least one box contains $\lceil \frac{n}{k} \rceil$ (or more) objects.

(b) at least one box contains $\lfloor \frac{n}{k} \rfloor$ (or fewer) objects.

Proof

The average number of objects per box is $\frac{n}{k}$.

We can't have all boxes below average. So some box must have $\geq \frac{n}{k}$ objects. but the # of objects in a box is an integer. so some box contains $\geq \lceil \frac{n}{k} \rceil$ objects, proving (a).

Similarly, not all boxes are above average. So some box contains $\le \frac{n}{k}$ objects, hence contains $\le \lfloor \frac{n}{k} \rfloor$ objects, proving (b). $\blacksquare$

<u>Rmks</u>

- If $n > k$, then $\lceil \frac{n}{k} \rceil \ge \frac{n}{k} > 1$.
  By (a) some box contains <u>2 or more objects</u>.
- If $n < k$, then $\lfloor \frac{n}{k} \rfloor \le \frac{n}{k} < 1$.
  By (b) some box contains <u>no objects</u>.

Ex.

find the smallest n such that in any group of n people, at least 10 were born in the same month.

objects : people n

boxes : months 12

By (a) of P.P. if n objects are placed in 12 boxes, some box contains $\lceil \frac{n}{12} \rceil$ (or more) objects.

we seek smallest n for which

$$\left\lceil \frac{n}{12} \right\rceil = 10$$

$$\therefore \quad 9 < \frac{n}{12} \leq 10$$

$$\therefore \quad 9 \cdot 12 < n \leq 10 \cdot 12 = 120$$

The smallest such n is

$$n = 9 \cdot 12 + 1 = 108 + 1 = \boxed{109}$$

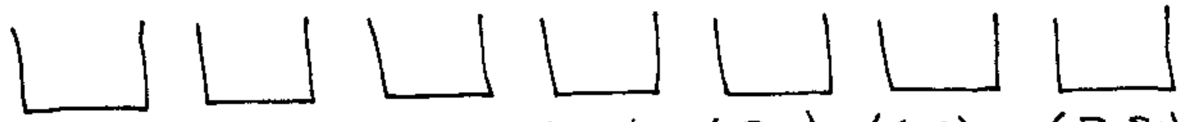
Ex.
suppose 8 numbers are chosen from $S = \{1, 2, 3, \ldots, 14\}$, show that at least 2 of the 8 numbers must add to 15.

note: we could check all $\binom{14}{8} = 3003$ 8-element subsets of S for the desired property.

objects : 8 numbers chosen

boxes : ?

form 7 labeled boxes



(1,14) (2,13) (3,12) (4,11) (5,10) (6,9) (7,8)

each of 8 chosen numbers is placed in the box with that number in the label. By P.P. (a) some box contains $\lceil \frac{8}{7} \rceil = 2$ numbers. those 2 numbers add to 15.

<u>Exercise</u>:

Generalize this example.

(3.10) combinatorial Proof

Ex. (3.6 #10)

show $\quad k \cdot \binom{n}{k} = n \binom{n-1}{k-1}$

Proof

Imagine we have n Players from which we must choose a team of size k, and one member of the team to be the captain.

we do this in 2 ways

I. (1) choose k Players from n : $\binom{n}{k}$

(2) choose 1 of k as captain : k

$\#\text{ways} = k \cdot \binom{n}{k}$

by the multiplication Principle.

II. (1) choose 1 of n as captain: n

(2) from remaining $(n-1)$ players, choose $(k-1)$ teammates: $\binom{n-1}{k-1}$

by mult. principle

$$\# \text{ ways} = n \cdot \binom{n-1}{k-1}$$

Since I & II count same things,

$$k\binom{n}{k} = n\binom{n-1}{k-1}. \quad \blacksquare$$

Rmk

Abstract ~~statement~~ of Problem. let S be a set with $|S| = n$. Both sides of formula count

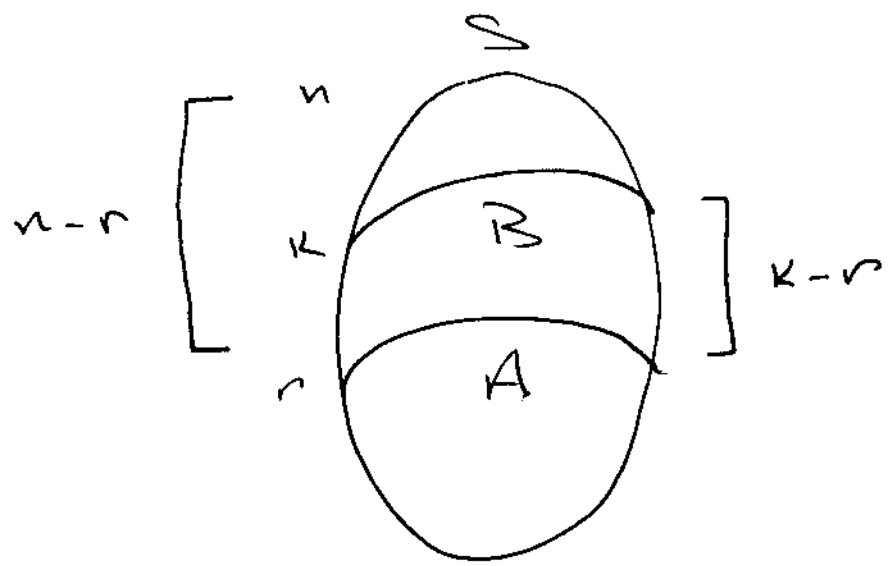
$$\{(x, A) \mid x \in A \subseteq S \text{ and } |A| = k\}$$

<u>Ex</u> show $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ $(0 \le r \le k \le n)$

count in 2 ways

$$\{(A,B) \mid A \subseteq B \subseteq S, |A| = r, |B| = k\}$$

for some set S with $|S| = n$.



I. (1) choose $B \subseteq S$: $\binom{n}{k}$

(2) choose $A \subseteq B$: $\binom{k}{r}$

$\#$ ways $= \binom{n}{k} \cdot \binom{k}{r}$

II. (1) choose $A \subseteq S$: $\binom{n}{r}$

(2) choose $B-A \subseteq S-A$: $\binom{n-r}{k-r}$

set $B = A \cup (B-A)$

$\#\text{ways} = \binom{n}{r}\binom{n-r}{k-r}$

since both I & II count same set.

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$$

<u>note</u>

1st example was a sp. case of this in which $A = \{x\}$, i.e. $r=1$. In this case $\binom{k}{r} = \binom{k}{1} = k$ and $\binom{n}{r} = \binom{n}{1} = n$. we get

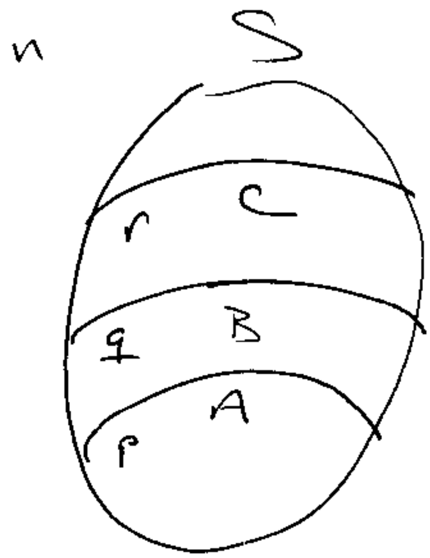
$$k \cdot \binom{n}{k} = n\binom{n-1}{k-1}.$$

Exercise

Generalize this identity by counting

$$\{(A,B,C) \mid A \subseteq B \subseteq C \subseteq S,\ |A|=p,\ |B|=q,\ |C|=r\}$$

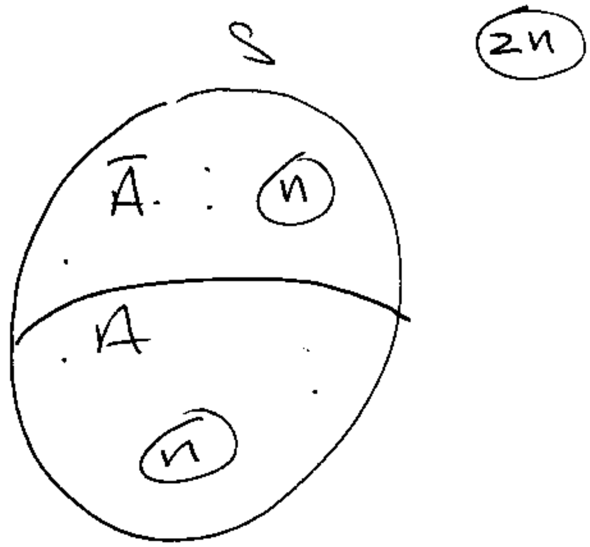
where S is a set with $|S|=n$



Ex. (3.10 #5)

show $\binom{2n}{2} = 2\binom{n}{2} + n^2$

let $|S| = 2n$ and $A \subseteq S$ with $|A| = n$.



LHS counts # of $\{x, y\} \subseteq S$.

RHS counts same thing. how?

we have 2 (mutually exclusive) choices

(1) • both x, y in A or both x, y in $\bar{A}$

(2) • x, y lie in differents i.e $x \in A$ and $y \in \bar{A}$ or $x \in \bar{A}$, $y \in A$.

\# ways of doing (1) : $\underbrace{2}_{\text{choose } A \text{ or } \overline{A}} \cdot \underbrace{\binom{n}{2}}_{\text{choose } x, y \in \text{the set}}$

\# ways of doing (2) : $\underbrace{n}_{\text{choose } x \in A} \cdot \underbrace{n}_{\text{choose } y \in \overline{A}} = n^2$

By sum rule

$$\binom{2n}{2} = 2\binom{n}{2} + n^2 \qquad \blacksquare$$