cse 16 7-14-20

Recall: Pascal's identity

for $0 < k < n$

$$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{①} + \underbrace{\binom{n-1}{k}}_{②}$$

①
$$\overbrace{\underbrace{b\ b\ b \cdots b}_{\substack{\text{len. } (n-1) \\ \text{with } (k-1)\ 1\text{'s}}}\ 1}^{\substack{\text{len. } n \\ \text{with } k\ 1\text{'s}}}$$

$\#\text{ways} = \binom{n-1}{k-1}$

②
$$\overbrace{\underbrace{b\ b\ b \cdots b}_{\substack{\text{len. } (n-1) \\ \text{with } k\ 1\text{'s}}}\ 0}^{\substack{\text{len. } n \\ \text{with } k\ 1\text{'s}}}$$

each $b \in \{0, 1\}$.

Pascal's Δ appears in another sequence of calculations

n	
0	$(x+y)^0 = 1$
1	$(x+y)^1 = 1\cdot x + 1\cdot y$
2	$(x+y)^2 = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$
3	$(x+y)^3 = 1\cdot x^3 + 3\cdot x^2 y + 3\cdot xy^2 + 1\cdot y^3$
4	$(x+y)^4 = 1\cdot x^4 + 4\cdot x^3 y + 6x^2y^2 + 4xy^3 + 1\cdot y^4$
$\vdots$	$\vdots$

$$\boxed{\text{The coefficient of } x^{n-k}y^k \text{ is } \binom{n}{k} \text{ for } k=0 \text{ to } k=n.}$$

Binomial Theorem: for all $n \geq 0$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

The algebraic proof will have to wait until we cover induction.

Combinatorial Proof

number the factors in $(x+y)^n$

$$(x+y)^n = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}\cdots\cdots\underset{n}{(x+y)}$$

before any like terms are combined, we have 2^n terms, each of form

$$x^{n-k}y^k$$

where $0 \leq k \leq n$

Illustrate with $n = 3$:

$$(x+y)^3 = \underset{①}{(x+y)}\underset{②}{(x+y)}\underset{③}{(x+y)}$$

$$\left.\begin{aligned} &= xxx + xxy + xyx + xyy \\ &\quad + yxx + yxy + yyx + yyy \end{aligned}\right\} \text{string over } \{x,y\} \text{ of len. } 3$$

$$\begin{aligned} &= x^3 + \underline{x^2y} + \underline{x^2y} + x.y^2 \\ &\quad + \underline{x^2y} + x.y^2 + x.y^2 + y^3 \end{aligned}$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Any terms with same # of x's and y's can be combined. So the coeff. of $x^{n-k}y^k$ is the number of strings over $\{x,y\}$ of length n containing exactly k y's, which is $\binom{n}{k}$.

Equivalently the coeff. of $x^{n-k} y^k$ is the # of ways of choosing which k of the positions $\{1, 2, 3, \ldots, n\}$ will contribute a y, i.e. $\binom{n}{k}$. Hence

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

<u>note</u>: numbers $\binom{n}{k}$ are sometimes called <u>binomial coefficients</u>

<u>Ex.</u> find coeff. of $x^8 y^5$ in $(x+y)^{13}$

ans. $\binom{13}{5} = \dfrac{13!}{5!\ 8!} = \dfrac{13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$

$= 13 \cdot 11 \cdot 9 = \boxed{1287}$

Ex. find coeff. of x^5 in $(x-1)^{10}$

$$(x-1)^{10} = \sum_{k=0}^{10} \binom{10}{k} \cdot (-1)^k x^{10-k}$$

$$10 - k = 5 \Rightarrow \boxed{k=5}$$

$$\text{Coef.} = (-1)^5 \binom{10}{5} = -\binom{10}{5} = \boxed{-252}$$

Ex. show $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

i.e. alternating sum of any row of Pascal's Δ is 0

e.g. $n=5$: $1-5+10-10+5-1 = 0$

$n=6$: $1-6+15-20+15-6+1 = 0$

Proof:

let $x = 1$ and $y = -1$ in Binomial Theorem

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot (-1)^k$$

$$\therefore \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 . \qquad \blacksquare$$

Ex. (3.6) #9

show $$9^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^{n-k}$$

Proof

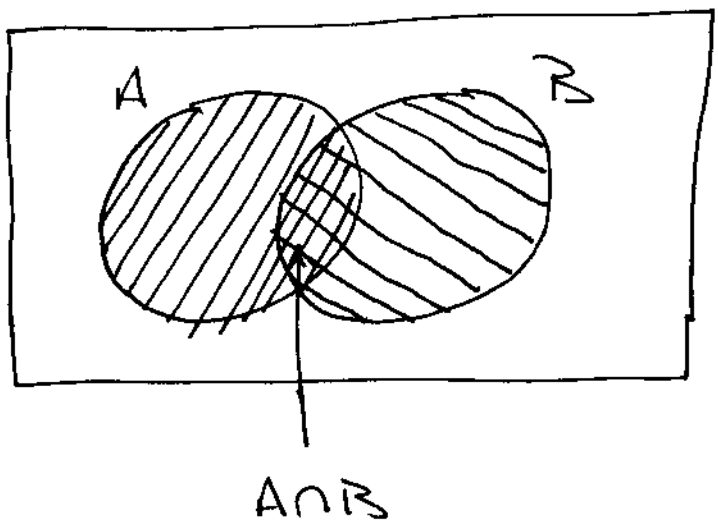
let $x = 10$ and $y = -1$ in Bin. Thm.

$$9^n = (10-1)^n = \sum_{k=0}^{n} \binom{n}{k} 10^{n-k} \cdot (-1)^k \qquad \blacksquare$$

(3.7) Principle of Inclusion-Exclusion (PIE)

Recall: if A, B finite, so are $A \cup B$ and $A \cap B$ and

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Ex. how many 5-card Poker hands are there? note $|\text{Deck}| = 52$.

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2} = \boxed{2{,}598{,}960}$$

Ex.

how many 5-card poker hands contain at least 1 red ace?

Poker deck:

Suits:

Kinds	red		black	
	hearts	diamonds	clubs	spades
	A	A	A	A
	2	⋮	⋮	⋮
	3			
	4			
	5			
	6			
	7			
	8			
	9			
	10			
	J			
	Q	⋮	⋮	⋮
	K	K	K	K

define

$A = \{$ 5-card P.H. with Ace of $\diamondsuit \}$

$B = \{$ " " " " " " Ace of $\heartsuit \}$

$A \cap B = \{$ " " " " " with both ace of $\diamondsuit$ & $\heartsuit \}$

$|A| = |B| = \binom{51}{4} = 249,900$

$|A \cap B| = \binom{50}{3} = 19,600$

so

$|\{$ 5-card P.H. with atleast 1 red ace $\}|$

$= |A \cup B| = |A| + |B| - |A \cap B|$

$= 480,200$

- PIE for 3 sets: A, B, C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- PIE for 4 sets: A, B, C, D

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|\end{aligned}$$

PIE for n sets:

$$A_1, A_2, A_3, \ldots, A_n$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

$$-\sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+\sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

$$\vdots$$

$$+(-1)^{n+1}\left|\bigcap_{k=1}^{n} A_k\right|$$

(3.8) Multisets

<u>Recall</u>: a <u>set</u> is an unordered collection with no repetition.

<u>notation</u>: $\{\dots\}$

a <u>list</u> is an ordered collection, with repetition allowed.

<u>notation</u>: $(\dots)$

<u>Defn</u>

an unordered collection in which multiple memberships are allowed, is called a <u>multi-set</u>.

<u>notation</u>: $[\dots]$

<u>Ex</u> $[1, 1, 1, 2, 3, 3, 4, 4, 4]$