cse 16 7-1-20

(2.6) logical equivalence

Defn
A tautology is a compound statement that is true for any assignment of truth values to its prop. vars.

Ex. $p \vee \sim p$ is a tautology

p	$\sim p$	$p \vee \sim p$
0	1	1
1	0	1

↑
all true!

Ex. $P \Rightarrow P$ (exercise)

Ex. $P \Rightarrow (P \vee Q)$

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

Defn

A contradiction is a compound proposition that is false for every assignmet of truth values to its prop. var.

Ex $P \wedge \sim P$

P	$\sim P$	$P \wedge \sim P$
0	1	0
1	0	0

↑
all false!

contradictions are said to be <u>unsatisfiable</u>. a statement that is not a contradiction is called <u>satisfiable</u>

<u>Ex</u>. P

<u>Defn</u>
A statement that is neither a contradiction nor a tautology is called a <u>contingency</u>.

<u>Defn</u>

Two compound statements X, Y are said to be <u>logically equivalent</u> if the statement

$$X \Longleftrightarrow Y$$

is a tautology. We write

$$X = Y$$

<u>Remarks</u>

- equiv. defn: $X = Y$ if their truth tables are identical.
- other symbols: $\equiv$, $\Longleftrightarrow$
- this is same meaning of $=$ sign in algebra: $a(b+c) = ab + ac$.

Ex. implication law

$$\underbrace{P \Rightarrow Q}_{①} \equiv \underbrace{\sim P \vee Q}_{②}$$

P	Q	P ⇒ Q	~P	~P ∨ Q	① ⇔ ②
				1	1
0	0	1	1	1	1
0	1	1	1	0	1
1	0	0	0	1	1
1	1	1	0		

Ex. contrapositive law

$$P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$$

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Ex. $P \Rightarrow Q \neq Q \Rightarrow P$ (converse)

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

different.

Exercise

use truth tables to prove all identities on P. 52 of [BOP]

- implication : $P \Rightarrow Q = \sim P \vee Q$
- contrapositive : $P \Rightarrow Q = \sim Q \Rightarrow \sim P$
- DeMorgan : $\sim(P \wedge Q) = \sim P \vee \sim Q$
  $\sim(P \vee Q) = \sim P \wedge \sim Q$
- commutative : $P \wedge Q = Q \wedge P$
  $P \vee Q = Q \vee P$
- Distributive : $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
- associative : $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$
  $P \vee (Q \vee R) = (P \vee Q) \vee R$

so we can write $P \wedge Q \wedge R$
$P \vee Q \vee R$

Ex 1st De Morgan: $\sim(P \wedge Q) = \sim P \vee \sim Q$

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

same

Exercise: Prove all of these.

we can use identities to prove other identities

Ex. $(P \vee Q) \Rightarrow R = (P \Rightarrow R) \wedge (Q \Rightarrow R)$

algebraic Proof:

$(P \vee Q) \Rightarrow R$

$= \sim(P \vee Q) \vee R$ (implication)

$= (\sim P \wedge \sim Q) \vee R$ (Demorgan)

$= (\sim P \vee R) \wedge (\sim Q \vee R)$ (comm. & dist.)

$= (P \Rightarrow R) \wedge (Q \Rightarrow R)$ (implication)

other identities:

- Domination: $P \vee T = T$
  $P \wedge F = F$

- Identity: $P \wedge T = P$
  $P \vee F = P$

- Idempotent: $P \vee P = P$

  $P \wedge P = P$

- Double negation: $\sim(\sim P) = P$

- Absorption: $P \vee (P \wedge Q) = P$

  $P \vee (P \wedge Q) = P$

- Negation: $P \vee \sim P = T$

  $P \wedge \sim P = F$

<u>Exercise</u>: Prove them all!

<u>Ex</u> show $(P \wedge Q) \Rightarrow (P \Rightarrow Q)$ is a tautology, algebraically.

Proof.

$$(P \wedge Q) \Rightarrow (P \Rightarrow Q)$$

$$= \sim(P \wedge Q) \vee (\sim P \vee Q) \quad \text{(implication)}$$

$$= (\sim P \vee \sim Q) \vee (\sim P \vee Q) \quad \text{(DeMorgan)}$$

$$= (\sim P \vee \sim P) \vee (\sim Q \vee Q) \quad \text{(comm., assoc.)}$$

$$= \sim P \vee T \quad \text{(Idem., Negation)}$$

$$= T \quad \text{(Domination}$$

Exercise

Show (algebraically) that

$$\sim (P \Rightarrow Q) \Rightarrow P$$

is a tautology

Exercise

Determine whether

$$(P \vee \sim Q) \wedge (Q \vee \sim R) \wedge (R \vee \sim P)$$

is satisfiable.

set theory and logic

let A, B, C be subsets of some universal set $\mathcal{U}$. look at definition

- $A \cap B = \{x \in \mathcal{U} \mid (x \in A) \wedge (x \in B)\}$
- $A \cup B = \{x \in \mathcal{U} \mid (x \in A) \vee (x \in B)\}$
- $A \oplus B = \{x \in \mathcal{U} \mid (x \in A) \oplus (x \in B)\}$
- $\overline{A} = \{x \in \mathcal{U} \mid x \notin A\} = \{x \in \mathcal{U} \mid \sim (x \in A)\}$
- $\phi = \{x \in \mathcal{U} \mid F\}$
- $\mathcal{U} = \{x \in \mathcal{U} \mid T\}$

analogy:

sets	$\longleftrightarrow$	statements
$\cup$	$\longleftrightarrow$	$\vee$
$\cap$	$\longleftrightarrow$	$\wedge$
$\oplus$	$\longleftrightarrow$	$\oplus$
$-$ (complement)	$\longleftrightarrow$	$\sim$ (negation)
ϕ	$\longleftrightarrow$	F
$\mathcal{U}$	$\longleftrightarrow$	T

<u>All</u> logical identities translate into set identities:

<u>Ex</u> Demorgan for sets

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Prove $\longrightarrow$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof:

$$\begin{aligned}
\overline{A \cup B} &= \{x \in U \mid x \notin A \cup B\} \\
&= \{x \in U \mid \sim(x \in A \cup B)\} \\
&= \{x \in U \mid \sim(x \in A \vee x \in B)\} \\
&= \{x \in U \mid \sim(x \in A) \wedge \sim(x \in B)\} \\
&= \{x \in U \mid x \in \overline{A} \wedge x \in \overline{B}\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}$$

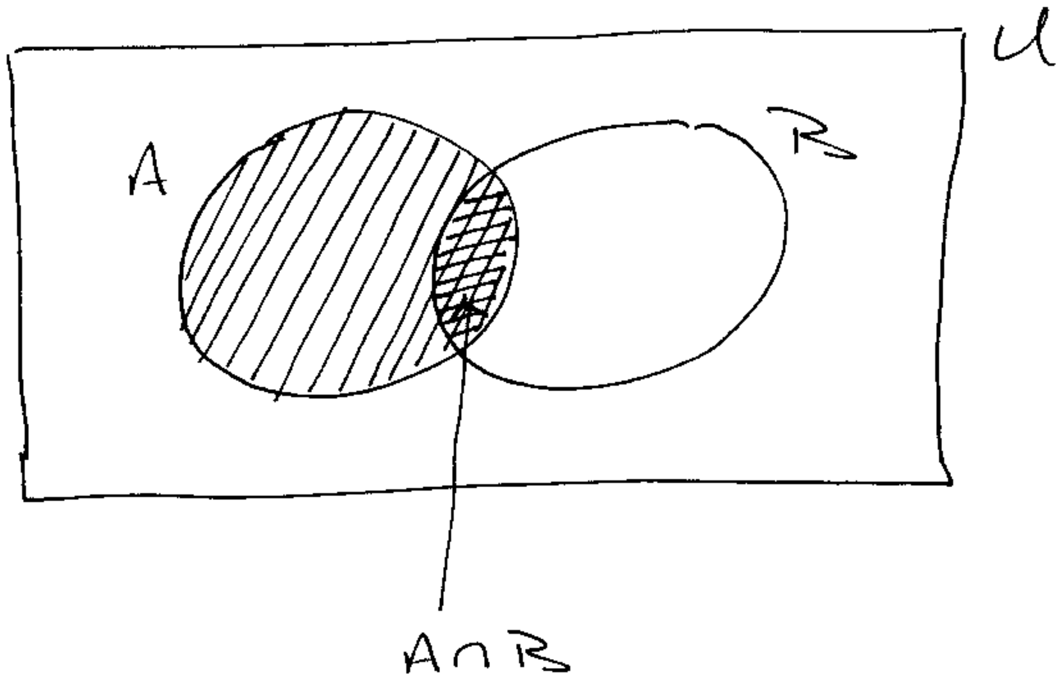
Exercise: Prove 1st Demorgan

Exercise:
state and prove all set identities corresponding to our logical identities

for instance, the absorption laws become

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

→ Venn diagram



Mathematical theory: Boolean Algebra