CMPE 16
Homework Assignment 5
Solutions to Selected Problems

1. (5.1) #6
   Prove that for any positive integer n:

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$$

Proof:

I. When $n = 1$ the above equation becomes $1 \cdot 1! = (1+1)! - 1$, i.e. $1 = 2 - 1$, i.e. $1 = 1$, which is true. The base case is therefore satisfied.

II. Let $n \geq 1$ be chosen arbitrarily. Assume, for this n, that

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$$

We must show that

$$\sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{n+1} k \cdot k! &= \left(\sum_{k=1}^{n} k \cdot k!\right) + (n+1) \cdot (n+1)! \\
&= \big((n+1)! - 1\big) + (n+1) \cdot (n+1)! \quad \text{by the induction hypothesis} \\
&= \big((n+1)! + (n+1) \cdot (n+1)!\big) - 1 \\
&= (n+1)! \cdot \big((n+1) + 1\big) - 1 \\
&= (n+1)! \cdot (n+2) - 1 \\
&= (n+2)! - 1.
\end{aligned}$$

The result now follows for all $n \geq 1$ by the Principle of Mathematical Induction. ■

2. <u>(5.1) #14</u>
Prove that for every positive integer n,

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

Proof:
I. For $n = 1$ the equation says $1 \cdot 2^1 = (1-1)2^{1+1} + 2$, i.e. $2 = 2$, which is true.

II. Let $n \geq 1$, and assume

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

We must show that

$$\sum_{k=1}^{n+1} k2^k = n2^{n+2} + 2.$$

We have

$$\sum_{k=1}^{n+1} k2^k = \left(\sum_{k=1}^{n} k2^k\right) + (n+1)2^{n+1}$$

$$= \big((n-1)2^{n+1} + 2\big) + (n+1)2^{n+1} \qquad \text{by the induction hypothesis}$$

$$= \big((n-1) + (n+1)\big)2^{n+1} + 2$$

$$= 2n \cdot 2^{n+1} + 2$$

$$= n2^{n+2} + 2,$$

as required. The result follows for all $n \geq 1$ by the PMI. ■

3. (5.3) #14

Let F_n denote the n^{th} Fibonacci number. Show that for all $n \geq 1$

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Proof:

I. If $n = 1$, then the formula says $F_2F_0 - F_1 = (-1)^1$, which reduces to $0 - 1 = -1$, which is true. The base case is therefore satisfied.

II. Let $n \geq 1$, and assume $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. We must show $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$.

Thus

$$\begin{aligned}
F_{n+2}F_n - F_{n+1}^2 &= F_{n+2}F_n - F_{n+1}F_{n+1} \\
&= (F_{n+1} + F_n)F_n - F_{n+1}(F_n + F_{n-1}) && \text{by the Fibonacci recurrence} \\
&= F_{n+1}F_n + F_n^2 - F_{n+1}F_n - F_{n+1}F_{n-1} \\
&= F_n^2 - F_{n+1}F_{n-1} \\
&= -(F_{n+1}F_{n-1} - F_n^2) \\
&= -(-1)^n && \text{by the induction hypothesis} \\
&= (-1)^{n+1}
\end{aligned}$$

The result follows for all $n \geq 1$ by the PMI. ■