CMPE 16
Homework Assignment 3
Solutions to Selected Problems

1. <u>(2.3) #38</u>
   Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c and d are constants. Determine necessary and sufficient conditions on the constants a, b, c and d so that $f \circ g = g \circ f$.

   Solution:
   Note $f \circ g = g \circ f$ means that $f \circ g(x) = g \circ f(x)$ for all $x \in \mathbb{R}$. We compute

   $$f \circ g(x) = f\big(g(x)\big) = f(cx + d) = a(cx + d) + b = acx + ad + b,$$

   and

   $$g \circ f(x) = g\big(f(x)\big) = g(ax + b) = c(ax + b) + d = acx + bc + d.$$

   Thus for $f \circ g = g \circ f$, it is necessary and sufficient that $\boldsymbol{ad + b = bc + d}$, which can also be written as $\boldsymbol{ad - bc = d - b}$. ■

2. <u>(2.4) #40</u>
   Determine the sum $\sum_{k=99}^{200} k^3$.

   Solution:
   Using the formula

   $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

   we have

   $$\sum_{k=99}^{200} k^3 = \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3$$

   $$= \left(\frac{200 \cdot 201}{2}\right)^2 - \left(\frac{98 \cdot 99}{2}\right)^2$$

   $$= (100 \cdot 201)^2 - (49 \cdot 99)^2$$

   $$= (20100\ )^2 - (4851)^2$$

   $$= 404{,}010{,}000 - \ 23{,}532{,}201$$

   $$= \mathbf{380{,}477{,}799}$$

   ■

3. (2.5) #20
Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof:
Assume $|A| = |B|$ and $|B| = |C|$. Then there exist bijective functions $f: A \to B$ and $g: B \to C$. It will be sufficient to show that the composition $g \circ f: A \to C$ is bijective, for then $|A| = |C|$.

(1) $g \circ f$ is injective: (This was proved in lecture from 7/12/23 on page 14 of the lecture notes.)

Let $x_1, x_2 \in A$, and suppose $g \circ f(x_1) = g \circ f(x_2)$. Being bijective, both g and f are injective, and hence

$$\begin{aligned} & g\big(f(x_1)\big) = g\big(f(x_2)\big) & \\ \therefore \quad & f(x_1) = f(x_2) & \text{since } g \text{ is injective} \\ \therefore \quad & x_1 = x_2 & \text{since } f \text{ is injective} \end{aligned}$$

If follows that $g \circ f$ is injective.

(2) $g \circ f$ is surjective:

Being bijective, both g and f are surjective. Let $z \in C$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. Then

$$g \circ f(x) = g\big(f(x)\big) = g(y) = z,$$

showing that $g \circ f$ is surjective.

Since $g \circ f: A \to C$ is both injective and surjective, it is bijective and hence $|A| = |C|$. ■

Alternate Proof:
Assume $|A| = |B|$ and $|B| = |C|$. Then there exist bijective functions $f: A \to B$ and $g: B \to C$. It will be sufficient to show that the composition $g \circ f: A \to C$ is bijective, for then $|A| = |C|$.

Since f and g are bijective, they are invertible (as was shown in lecture on 7/12/23) with inverses $g^{-1}: C \to B$ and $f^{-1}: B \to A$, respectively. Using the associativity of composition, we have

$$\begin{aligned} (g \circ f) \circ (f^{-1} \circ g^{-1}) &= \big(g \circ (f \circ f^{-1})\big) \circ g^{-1} \\ &= (g \circ i_B) \circ g^{-1} \\ &= g \circ g^{-1} \\ &= i_C, \end{aligned}$$

and

$$\begin{aligned} (f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ \big((g^{-1} \circ g) \circ f\big) \\ &= f^{-1} \circ (i_B \circ f) \\ &= f^{-1} \circ f \\ &= i_A. \end{aligned}$$

Thus $g \circ f$ is invertible with inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Being invertible, $g \circ f: A \to C$ is bijective, and hence $|A| = |C|$. ■