CSE 16   6-6-24

Review Problems from 8.1

#9-14, 24-27

10 a

Find a recurrence for the # of bit strings of len. n containing '01'

Let B_n = # bit strs of len n containing '01'

classes of strings: # ways

(1) $\overset{\text{'01'}}{\underbrace{x\,x\cdots\cdots x}_{n-1}}\,0$: B_{n-1}

(2) $\overset{\text{any}}{\underbrace{x\cdots\cdots x}_{n-2}}\,01$: 2^{n-2}

(3) $\overset{\text{any}}{\underbrace{x\cdots\cdots x}_{n-3}}\,011$: 2^{n-3}

(4) $\overset{\text{any}}{\underbrace{x\cdots\cdots x}_{n-4}}\,0111$: 2^{n-4}

$\vdots$ $\vdots$

(n) $0111\cdots\cdots 1$: $2^{n-n} = 2^0 = 1$

By the sum rule

$$B_n = B_{n-1} + \sum_{k=0}^{n-2} 2^k$$

$$= B_{n-1} + \frac{2^{n-1} - 1}{2 - 1}$$

$$\therefore \begin{cases} B_n = B_{n-1} + 2^{n-1} - 1 \\ B_0 = 0 \qquad \checkmark \end{cases}$$

- check that $\boxed{B_n = 2^n - n - 1}$

Solve the above recurrence.

check!

$$\text{RHS} = B_{n-1} + 2^{n-1} - 1$$

$$= \left(2^{n-1} - (n-1) - 1\right) + 2^{n-1} - 1$$

$$= 2^{n-1} + 2^{n-1} - n + \cancel{1} - \cancel{1} - 1$$

$$= 2^{1} \cdot 2^{n-1} - n - 1$$

$$= 2^{n} - n - 1$$

$$= B_n = \text{LHS}.$$

Another Way.

Let A_n = # bit strings of len. n that do not contain '01'

so $A_n + B_n = 2^n$

If a bit string does not contain '01', then all 1's occur to the left of all 0's

$$\left.\begin{array}{l} 0 \cdots\cdots 0 \\ 1\ 0 \cdots\cdots 0 \\ 1\ 1\ 0 \cdots 0 \\ 1\ 1\ 1\ 0 \cdots 0 \\ \vdots \\ 1\ 1\ 1 \cdots 1\ 0 \\ 1\ 1\ 1 \cdots 1\ 1 \end{array}\right\} A_n = n+1$$

$\therefore (n+1) + B_n = 2^n$

$$\therefore \boxed{B_n = 2^n - n - 1}$$

#24

find a recurrence for the # of bit strings of len. n containing an even # of 0's.

Let E_n = # bit strs. of len. n with an even # of 0's

Then $(2^n - E_n)$ = # bit strs. of len. n with odd # of 0's.

classes of strings :

<u>#ways</u>

(1) $\overset{\text{even}}{\underbrace{x\cdots\cdots x}_{n-1}}\,1$: E_{n-1}

(2) $\overset{\text{odd}}{\underbrace{x\cdots\cdots x}_{n-2}}\,10$: $2^{n-2}-E_{n-2}$

(3) $\overset{\text{even}}{\underbrace{x\cdots\cdots x}_{n-2}}\,00$: E_{n-2}

By the sum rule

$$E_n = E_{n-1} + \left(2^{n-2} - \cancel{E_{n-2}}\right) + \cancel{E_{n-2}}$$

$$\therefore \left\{ \begin{array}{l} E_n = E_{n-1} + 2^{n-2} \\ E_1 = 1 \end{array} \right.$$

- Check $\boxed{E_n = 2^{n-1}}$ solves this relation.

Another way!

classes of strings

(1) $\overset{\text{even } 0\text{'s}}{\underbrace{x \cdots\cdots x}_{n-1}}\; 1$: E_{n-1}

(2) $\overset{\text{odd } 0\text{'s}}{\underbrace{x \cdots\cdots x}}\; 0$: $\left(2^{n-1} - E_{n-1}\right)$

By sum rule!

$$E_n = \cancel{E_{n-1}} + \left(2^{n-1} - \cancel{E_{n-1}}\right)$$

$$\therefore \boxed{E_n = 2^{n-1}}$$

Ex. what is the prob. that a randomly chosen integer in range 1 to 100 is divisible by 7 ?

$S = \{1, 2, \ldots, 100\}, \quad |S| = 100$

$E = \{7, 14, 21, \ldots, 98\} \quad |E| = \left\lfloor \frac{100}{7} \right\rfloor = 14$

ans. $= \frac{|E|}{|S|} = \frac{14}{100} = \frac{7}{50} = \boxed{.14}$