CSE 16 6-4-24

Final: Mon 6/10. 9:30-11:00 am

Quiz4: Thur 6/6

SETs: Due Sun 6/9. 11:59 pm

If response rate $\geq 85\%$, then will add 0.5% to final score.

Ex.

a single fair die is cast and we observe whether

result ≤ 4 or result ≥ 5

result : $\underbrace{1 \quad 2 \quad 3 \quad 4}_{\text{success}}$ $\underbrace{5 \quad 6}_{\text{failure}}$

$p = \frac{4}{6} = \boxed{\frac{2}{3}}$ $\qquad q = \boxed{\frac{1}{3}}$

Repeat the experiment 6 times. What is the Prob. that half of the tosses are ≤ 4 ?

We have $n=6$, $k=3$. Thus

$$\text{Prob.} = \binom{6}{3} \cdot \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3$$

$$= \frac{160}{729} = .2195\ldots$$

Ex. 7.2 # 35 a, b, c, d

a) Prob. of 0 failures in n ind. Bernoulli trials.

answer $= \boxed{p^n}$

d)

P(at least 2 failures)

$= P(\# \text{fail} \geq 2)$

$= 1 - P(\# \text{fail} \leq 1)$

$= 1 - (p^n + n q p^{n-1})$

$= \boxed{1 - p^n - n p^{n-1}(1-p)}$

<u>Exercise</u> 7.2 #36

8.1 Applications of Recurrences

Ex

Let B_n denote the # of bit strings of length n that contain the substring '00'.

We find a recurrence relation

$$\left(B_n\right)_{n=0}^{\infty}$$

note: $B_0 = B_1 = 0$, $B_2 = 1$,

calculate $B_3 = 3$

000 ~~010~~ 100 ~~110~~
001 ~~011~~ ~~101~~ ~~111~~

We can construct a bit str. of len n containing '00' by performing exactly one of the following subtasks

(1) form a bit-str. of len $n-1$ containing '00', append 1.

'00' $\underbrace{x\,x \cdots\cdots x}_{n-1}\ 1$ #ways $= B_{n-1}$

(2) form a bit-str. of len $n-2$ cont. '00', append '10'

'00' $\underbrace{x\,x \cdots\cdots x}_{n-2}\ 1\,0$ #ways $= B_{n-2}$

(3) form an arbitrary bit-str. of len. $n-2$, append '00'

$$\underbrace{\overset{\text{any}}{x\,x \cdots x}}_{n-2}\,00 \qquad \#\text{ways} = 2^{n-2}$$

By the sum rule

$$\begin{cases} B_0 = 0 \\ B_1 = 0 \\ B_n = B_{n-1} + B_{n-2} + 2^{n-2} \quad (n \geq 2) \end{cases}$$

$$\therefore \quad B_2 = 0 + 0 + 2^0 = 1 \checkmark$$

$$B_3 = 1 + 0 + 2^1 = 3 \checkmark$$

$$B_4 = 3 + 1 + 2^2 = 8$$

$$B_5 = 8 + 3 + 2^3 = 19$$

$$B_6 = 19 + 8 + 2^4 = 43$$

- Find the Prob. that a random bit string of len. 6 Contains '00'

$|S| = 2^6 = 64$

$|E| = 43$

$\text{Prob} = \frac{43}{64} = .6719$

<u>Exercise</u>

Show that

$$B_n = \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$$

<u>Hint</u>. $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ are roots of $x^2 = x + 1$

<u>Ex.</u> $ans = C_n$

find a recurrence for # bit strs of len. n containing '000'.

		# ways
(1)	$\underbrace{x\,x \cdots x}_{n-1}\,1$! ← 000	C_{n-1}
(2)	$\underbrace{x\,x \cdots x}_{n-2}\,10$! ← 000	C_{n-2}
(3)	$\underbrace{x\,x \cdots x}_{n-3}\,100$! ← 000	C_{n-3}
(4)	$\underbrace{x\,x \cdots x}_{n-3}\,000$! ← any	2^{n-3}

$$\therefore \begin{cases} C_0 = C_1 = C_2 = 0 \\ C_n = C_{n-1} + C_{n-2} + C_{n-3} + 2^{n-3} \end{cases}$$

so

$$C_3 = 0+0+0+2^0 = 1$$

$$C_4 = 1+0+0+2^1 = 3$$

$$C_5 = 3+1+0+2^2 = 8$$

$$C_6 = 8+3+1+2^3 = 20$$

$$\vdots$$

<u>Exercise</u>

find a recurrence for # of ternary strings (alphabet $= \{0, 1, 2\}$) of len. n containing '00'.

<u>Answer</u>

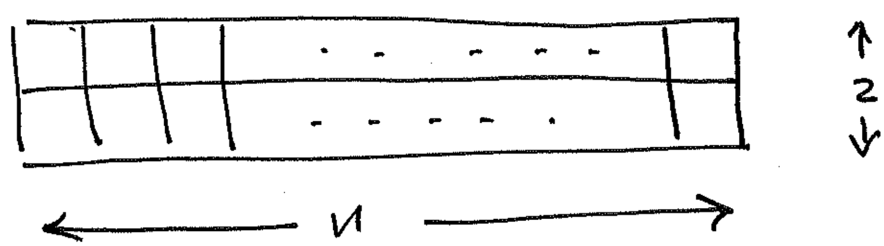
$$T_0 = T_1 = 0$$

$$T_n = 2T_{n-1} + 2T_{n-2} + 3^{n-2}$$

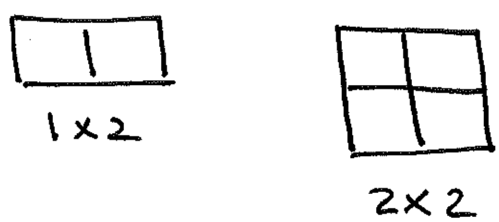
Ex.

In how many ways can we tile a $2 \times n$ rectangular walkway with 1×2 and 2×2 tiles?

walk:

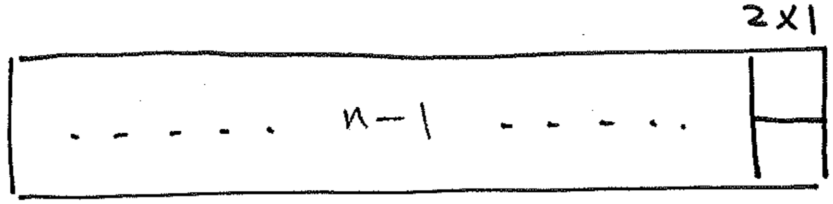


tiles:

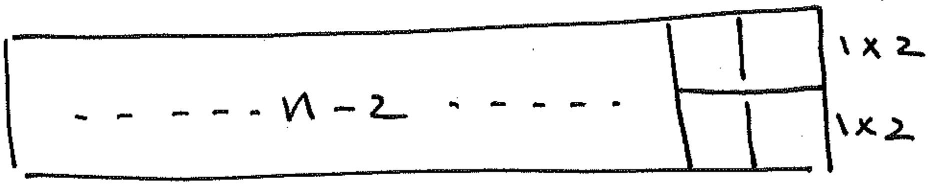


Let A_n = # of such tilings, $(n \geq 0)$

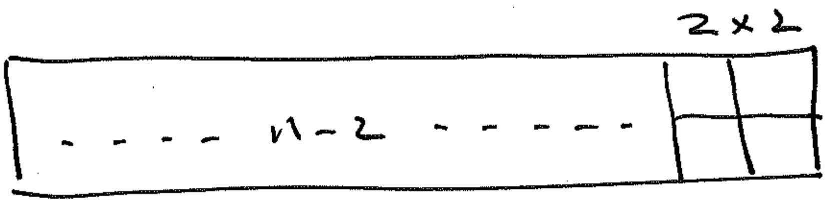
ways

(1) 

A_{n-1}

(2) 

A_{n-2}

(3) 

A_{n-2}

By sum rule: $A_n = A_{n-1} + 2A_{n-2}$ $(n \geq 2)$

Initial terms

$A_0 = 1$ (empty tiling)

$A_1 = 1$ ()

so

$$A_2 = 1 + 2\cdot 1 = 3$$ ()

$$A_3 = 3 + 2\cdot 1 = 5$$

$\vdots$

- find $A_8 = 171$
- find $A_{10} = 683$
- find Prob. that a random tiling of len. 10 ends in 2x2 

$$\text{ans} = \frac{171}{683} = .2504$$

- Check that: $\boxed{A_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)}$

Exercise 7.2 #36

show if $E_1, E_2, \ldots, E_n$ are Pairwise disjoint events in S, then

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Proof

I. for $n=1$, formula says

$$P(E_1) = P(E_1). \checkmark$$

II let $n \geq 1$. Assume

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

must show

$$P\left(\bigcup_{i=1}^{n+1} E_i\right) = \sum_{i=1}^{n+1} P(E_i)$$

so

$$P\left(\bigcup_{i=1}^{n+1} E_i\right) = P\left(\left(\bigcup_{i=1}^{n} E_i\right) \cup E_{n+1}\right)$$

$$= P\left(\bigcup_{i=1}^{n} E_i\right) + P(E_{n+1}) \quad \Big\{ \text{by axiom 3}$$

$$= \sum_{i=1}^{n} P(E_i) + P(E_{n+1}) \quad \Big\{ \text{by ind. hyp.}$$

$$= \sum_{i=1}^{n+1} P(E_i).$$

$\blacksquare$