CSE 16 5-28-24

Defn

Let $n = n_1 + n_2 + \cdots + n_k$ where each $n_i \in \mathbb{N}$ $(1 \le i \le k)$. The <u>multinomial coefficient</u>

$$\binom{n}{n_1, n_2, \ldots, n_k}$$

is the number of permutations of the n-element multiset

$$\{x_1^{n_1}, x_2^{n_2}, \ldots, x_k^{n_k}\}$$

i.e. anagrams of the string

$$\underbrace{x_1 \cdots x_1}_{n_1} \underbrace{x_2 \cdots x_2}_{n_2} \cdots\cdots \underbrace{x_k \cdots x_k}_{n_k}$$

By the last theorem:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

<u>Observe</u>:

$$\binom{n}{k, n-k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

and

$$\binom{n}{a, b, n-a-b} = \binom{n}{a}\binom{n-a}{b}$$

Multinomial Theorem

Let $x_1, x_2, \ldots, x_k \in \mathbb{R}$. Then

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{\text{all } (n_1, n_2, \ldots, n_k) \\ \text{s.t. } n_1 + n_2 + \cdots + n_k = n}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where the sum is over all k-tuples $(n_1, \ldots, n_k)$ s.t. $n_1 + \cdots + n_k = n$, and $n_i \in \mathbb{N}$ $(1 \leq i \leq k)$.

Exercise

Prove this by imitating the (combinatorial) Proof of Binomial Thm.

7.1 Intro to Probability

Defn

- An *experiment* is a procedure that that yields one of a set of possible *outcomes*.
- The set S of outcomes is called the *sample space* for the experiment.
- An *event* is a subset $E \subseteq S$.

- Let S be a finite sample space. A <u>Probability function</u>

$$P : \mathbb{P}(S) \to \mathbb{R}$$

satisfying certain axioms (7.2)

In this section, we concentrate on one Prob. fcn.

$$P(E) = \frac{|E|}{|S|}.$$

called the <u>uniform</u> Prob. law.

for any event $E \subseteq S$.

Observe:

(1) for any $E \subseteq S$: $0 \le P(E) \le 1$

(2) $P(\phi) = 0$

(3) $P(S) = 1$

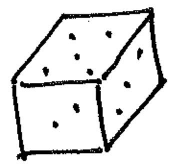
(4) if $E_1, E_2, \ldots, E_k$ are pairwise disjoint events in S, then

$$P(E_1 \cup \ldots \cup E_k) = P(E_1) + \cdots + P(E_k)$$

Exercise Prove (1)-(4) using defn of $P(E) = \frac{|E|}{|S|}$.

Ex.

A single die is thrown.



what is the prob. that the #
on top is at least 5 ?

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$E = \{5, 6\}$$

$$P(E) = \frac{2}{6} = \frac{1}{3} = .333\ldots$$

Ex.

Two dice are thrown. What is the Prob. that the sum is 8?

sum = 8

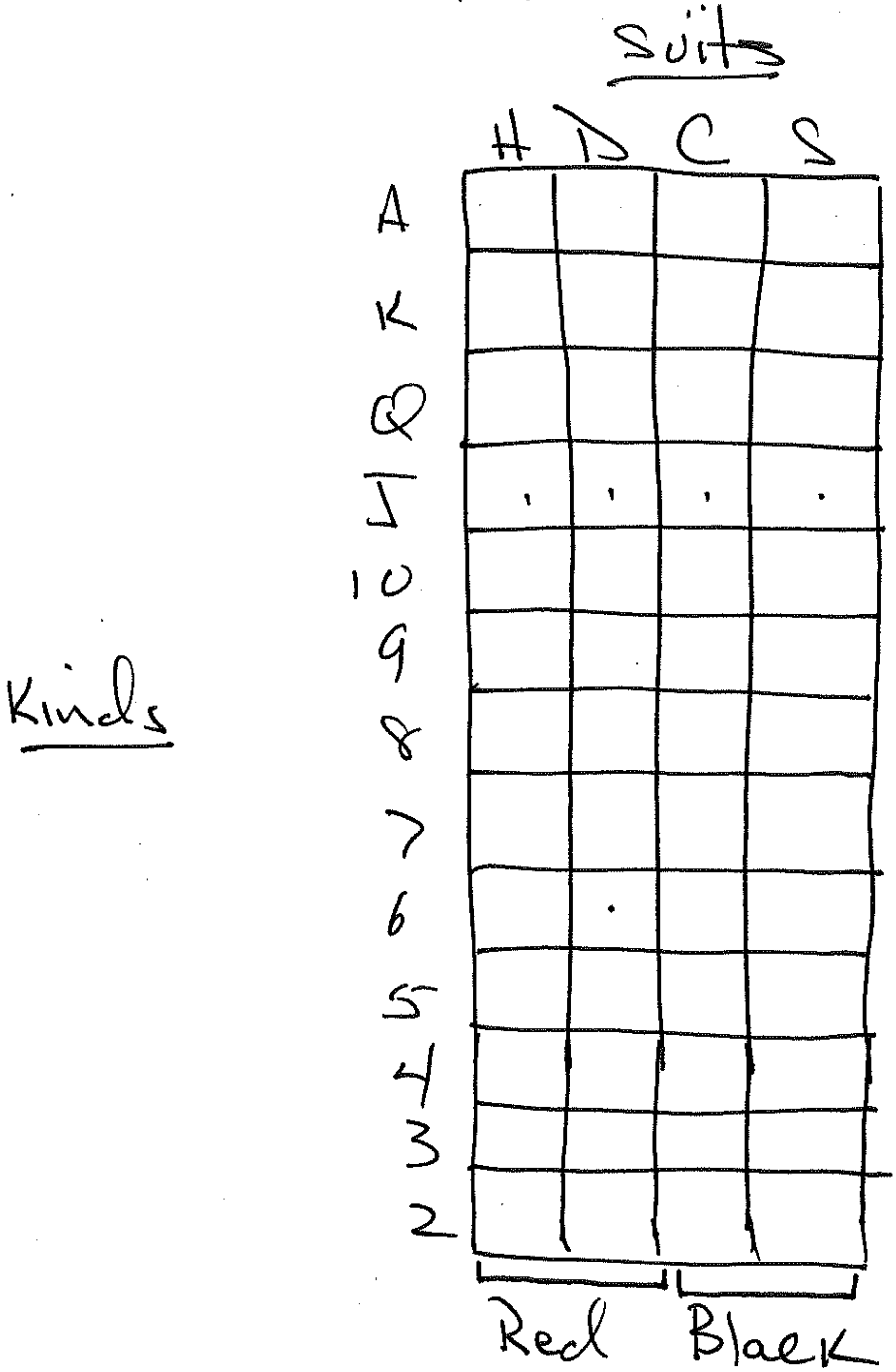
$$S = \left\{ \begin{matrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{matrix} \right\}$$

$E = \{62, 53, 44, 35, 26\}$

$$P(E) = \frac{5}{36} = .1388\ldots$$

Ex.

what is the prob. that a 5-card poker hand contains 4 of a kind.



#cards = $\boxed{52}$

$S = \{\text{all 5-card Poker hands}\}$

$$|S| = \binom{52}{5} = 2{,}598{,}960$$

$E = \{\text{hands containing 4 of a kind}\}$

$$|E| = \binom{13}{1} \cdot \binom{4}{4} \cdot \binom{48}{1} = 13 \cdot 1 \cdot 48 = 624$$

$$P(E) = \frac{624}{2{,}598{,}960} = \frac{1}{4165} \doteq .00024$$

Ex. What is prob. that a 5-card Poker hand has 3 of a kind (not 4 of a kind)

$$|E| = \binom{13}{1} \cdot \binom{4}{3}\binom{48}{2} = 58{,}656$$

$$P(E) = \frac{94}{4165} = .02256\ldots$$

Theorem

Given $E \subseteq S$, the prob. of $\overline{E} = S - E$ is

$$P(\overline{E}) = 1 - P(E)$$

also

$$P(E) = 1 - P(\overline{E})$$

Proof

note $E \cap \overline{E} = \phi$ and $E \cup \overline{E} = S$, so

$$1 = P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E})$$

$\blacksquare$

Ex.

What is prob. that a 5-card Poker hand contains at least one Ace?

$$E = \{\text{hands with } \geq 1 \text{ ace}\}$$

$$\bar{E} = \{\text{hands with } 0 \text{ aces}\}$$

$$|\bar{E}| = \binom{48}{5} = 1{,}712{,}304$$

$$\therefore P(E) = 1 - P(\bar{E}) = 1 - \frac{1712304}{2598960}$$

$$= .34115\ldots$$

Theorem

Given $E_1, E_2 \subseteq S$:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof

Divide

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

by $|S|$. $\blacksquare$

Ex. what is the prob. that a 5-card poker hand contains a red ace?

$E_1 = \{\text{hand contains } AH\}$

$E_2 = \{\text{ " \quad " } \quad AD\}$

$E_1 \cap E_2 = \{\text{ " \quad " both } AH \text{ \& } AD\}$

$|E_1| = \binom{51}{4} = |E_2|$

$|E_1 \cap E_2| = \binom{50}{3}$

$$P(E) = P(E_1 \cup E_2) = \frac{\binom{51}{4} + \binom{51}{4} - \binom{50}{3}}{\binom{52}{5}}$$

$$= .18476\ldots$$

Ex. To win a super lottery, Pick 7 correct numbers out of 40.

$$|S| = \binom{40}{7}, \quad |E| = 1$$

$$\therefore P(E) = \frac{1}{\binom{40}{7}} = (5.36..)\cdot 10^{-8}$$

In general

$$P(\text{win}) = \frac{\#\text{ Ways of Winning}}{\#\text{ Ways of Playing}}$$

Ex. another state lottery commision picks 12 numbers from $\{1, 2, \ldots, 60\}$. To win you must pick 6 of these 12 numbers.

$$\therefore P(\text{win}) = \frac{\binom{12}{6}}{\binom{60}{6}} = (1.85) \cdot 10^{-5}$$

<u>Ex.</u> find the Probability that a random string of len. 10 from $\{a, b, \ldots, z\}$ has no repeated letters.

$S = \{$strs. of len $= 10$ from $\{a, .., z\}\}$

$|S| = 26^{10}$

$|E| = P(26, 10) = \dfrac{26!}{16!}$

$\therefore P(E) = \dfrac{26!}{16! \cdot 26^{10}} = .1365\ldots$