CSE 16 5-21-24

Recorded Lecture

6.4 Binomial Coefficients

we'll use $\binom{n}{k}$ notation.

Recall: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ $(0 \le k \le n)$

observe:

$n=0$: $\binom{0}{0}=1$

$n=1$: $\binom{1}{0}=1$, $\binom{1}{1}=1$

$n=2$: $\binom{2}{0}=1$, $\binom{2}{1}=2$, $\binom{2}{2}=1$

$n=3$: $\binom{3}{0}=1$, $\binom{3}{1}=3$, $\binom{3}{2}=3$, $\binom{3}{3}=1$

$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
$\vdots$	$\vdots$ $\ddots$

This is Pascal's Triangle.

<u>Theorem</u> (Pascal's Identity)

(1) $$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad (1 \le k \le n)$$

alternate form

(2) $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (1 \le k < n)$$

<u>Proof of (1)</u> : (algebraic)

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-(k-1))!}$$

$$= \frac{n!\,(n-k+1)}{k!\,(n-k+1)!} + \frac{n!\,k}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(n-\cancel{k}+1+\cancel{k})}{k!\,((n+1)-k)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!}$$

$$= \binom{n+1}{k} \qquad \blacksquare$$

<u>Proof of (1)</u> (combinatorial)

Let T be a set with $|T| = n+1$. Let $x \in T$, and let $S = T - \{x\}$, so $|S| = n$.

The task of choosing a k-subset of T breaks into 2 subtasks, exactly one of which will be performed.

- include x : #ways $= \binom{n}{k-1}$
  choose a $(k-1)$-subset of S, then add x.
- exclude x : #ways $= \binom{n}{k}$
  choose a k-subset of S.

By the sum rule, the # of k-subsets of T is $\binom{n}{k} + \binom{n}{k-1}$. Therefore

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$\blacksquare$

Theorem

For all $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof (combinatorial)

Both sides count the # of subsets of an n-element set. $\blacksquare$

The Binomial Theorem

observe

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1 \cdot x + 1 \cdot y$$
$$(x+y)^2 = 1 \cdot x^2 + 2xy + 1 \cdot y^2$$
$$(x+y)^3 = 1 \cdot x^3 + 3x^2y + 3xy^2 + 1 \cdot y^3$$
$$(x+y)^4 = 1 \cdot x^4 + 4 \cdot x^3 y + 6 \cdot x^2 y^2 + 4 \cdot xy^3 + 1 \cdot y^4$$
$$\vdots$$

Each term in $(x+y)^n$ is of the form $x^{n-k}y^k$, where $0 \le k \le n$, with coefficients from Pascal's triangle.

<u>Theorem</u> (Binomial theorem)

Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

<u>Ex.</u> $n = 3$

$$(x+y)^3 = \underset{1}{(x+y)}\underset{2}{(x+y)}\underset{3}{(x+y)}$$

set	bit string	monomial	
$\emptyset$	000	$xxx = x^3$	x^3
$\{3\}$	001	$xxy = x^2y$	
$\{2\}$	010	$xyx = x^2y$	$3x^2y$
$\{2,3\}$	011	$xyy = xy^2$	
$\{1\}$	100	$yxx = x^2y$	
$\{1,3\}$	101	$yxy = xy^2$	$3xy^2$
$\{1,2\}$	110	$yyx = xy^2$	
$\{1,2,3\}$	111	$yyy = y^3$	y^3

after collecting like terms

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Proof (combinatorial)

when

$$(*) \qquad (x+y)^n = \overset{1}{(x+y)}\overset{2}{(x+y)} \cdots \overset{n}{(x+y)}$$

is expanded, and before like terms are combined, we have 2^n terms, each of the form

$$x^{n-k}y^k \quad (\text{for some } 0 \le k \le n)$$

Fix a particular k. To determine the coefficient of $x^{n-k}y^k$ (after like terms are combined) we must count the # of times this term appears in the expansion.

To obtain $x^{n-k}y^k$ we must select y from k factors in (*) and x from the remaining $n-k$ factors.

To choose which k of the n factors contribute y, is to choose a k-subset of the n-set $\{1, 2, \ldots, n\}$. So

$$\#\text{ways} = \binom{n}{k}$$

$\therefore$ $x^{n-k}y^k$ occurs $\binom{n}{k}$ times in the full expansion, so after combining like terms its coefficient is $\binom{n}{k}$.

Thus

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

$\blacksquare$

Corollary

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof

Let $x = y = 1$ in the Binomial Theorem.

∎

Corollary.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \qquad (\text{for } n \geq 1)$$

Proof

Let $x = 1, y = -1$ in Bin. Thm.

∎

The algebraic proof uses Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Proof (algebraic)

we show

$P(n)$

$$\forall n \geq 0 : \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

I. $P(0)$ says $(x+y)^0 = \binom{0}{0} x^0 y^0$, i.e. $1 = 1$, so base case is true.

IIa. $\forall n \geq 0 : P(n) \rightarrow P(n+1)$

Let $n \geq 0$. Assume

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

we must show

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k .$$

observe

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

$$= (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}}_{k \leftarrow k-1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-(k-1)} y^{(k-1)+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n+1-k} y^k$$

$$= \binom{n}{0} x^{n+1} y^0 + \left[\sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^k \right]$$

$$+ \binom{n}{n} x^0 y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^{n+1-k} y^k + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k .$$

Result follows by P.M.I. $\blacksquare$