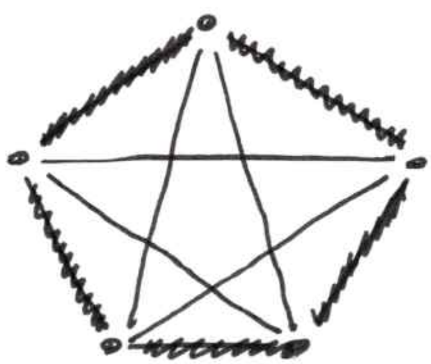
CSE 16 5-17-24

Supplemental

continuing 6.2 ....

<u>Note</u> 6 is the smallest number of people the conclusion is true

<u>Ex</u> $n = 5$



start of <u>Ramsey Theory</u>.

6.3 Permutations & Combinations

Defn

A Permutation of a set is an ordered arrangement of its elements.

Ex. $S = \{1, 2, 3\}$

Permutations of S:

123, 132, 213, 231, 312, 321

Defn

A K-Permutation of a set S is an ordered arrangement of K of its elements. $(0 \le K \le |S|)$.

Ex. $S = \{1, 2, 3\}$ $\quad k = 0, 1, 2, 3$

	k-Permutations of S	$P(3,k)$
$k=0$:	$\emptyset$	1
$k=1$:	1, 2, 3	3
$k=2$:	12, 21, 13, 31, 23, 32	6
$k=3$:	123, 132, 213, 231, 312, 321	6

Notation

$P(n,k) = \#$ of k-Permutations of an n-element set $(0 \le k \le n)$.

Theorem

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

for $0 \le k \le n$

Follows by the product rule.

Note

if $|S| = n$, then an n-Permutation of S is just a Permutation of S.

$$P(n,n) = n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1 = n!$$

Defn

A k-combination of S is an un-ordered collection of k of its elements, i.e. a k-element subset of S. $(0 \le k \le |S|.)$

<u>Ex.</u> $S = \{1, 2, 3\}$ $\quad k = 0, 1, 2, 3$

	<u>k-combinations of S</u>	<u>$C(3,k)$</u>
$k=0$:	ϕ	1
$k=1$:	$\{1\}, \{2\}, \{3\}$	3
$k=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	3
$k=3$:	$\{1,2,3\}$	1

<u>Notation</u>

- $C(n,k)$ = # k-combinations of an n-element set
- $\binom{n}{k}$ read: "n-choose-k"
- Called "Binomial Coefficients"

Theorem

for $0 \le k \le n$

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Proof.

Let S be a set with $|S| = n$.

Consider the task of constructing a k-permutation of S. We decompose into two subtasks, performed in succession.

- choose a k-subset of S.
  $\#$ways $= C(n,k)$.
- choose a permutation of these k elements. $\#$ways $= k!$

By the Product rule

$$P(n,k) = C(n,k) \cdot k!$$

$$\therefore \quad \frac{n!}{(n-k)!} = C(n,k) \cdot k!$$

$$\therefore \quad C(n,k) = \frac{n!}{k!(n-k)!}. \qquad \blacksquare$$

<u>Ex.</u>
How many bit strings of length 10 contain exactly 6 1's ?

$$\underset{1}{\underline{?}} \; \underset{2}{\underline{?}} \; \underset{3}{\underline{?}} \; \underset{4}{\underline{\;\;}} \; \underset{5}{\underline{\;\;}} \; \underset{6}{\underline{\;\;}} \; \underset{7}{\underline{\;\;}} \; \underset{8}{\underline{\;\;}} \; \underset{9}{\underline{\;\;}} \; \underset{10}{\underline{?}}$$

we must choose which 6 of the 10 Positions $\{1, 2, 3, \ldots, 10\}$ will be occupied by 1's. The remaining Positions are occupied by 0's.

answer

$$C(10,6) = \binom{10}{6} = \frac{10!}{6!\,4!}$$

$$= \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \cancel{8} \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$

$$= \boxed{210}$$

we could have chosen which 4 of the 10 positions will be occupied by o's.

$$C(10,4) = \frac{10!}{4!\,6!} = \boxed{210}$$

Theorem

for $0 \le k \le n$:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof (algebraic)

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(\cancel{n}-(\cancel{n}-k))!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}. \quad \blacksquare$$

Most identities have 2 types of proofs

- algebraic

  manipulate known identities

- combinatorial

  Argue that LHS and RHS count the same set.

Alternate Proof (combinatorial)

Let $|S| = n$, define for $0 \le k \le n$

$$S^{(k)} = \{ k\text{-subsets of } S \} \subseteq \mathbb{P}(S)$$

Define $f : S^{(k)} \to S^{(n-k)}$ by

$$f(A) = \bar{A} = S - A$$

Observe $f \circ f(A) = f(f(A)) = f(\bar{A})$
$= \bar{\bar{A}} = A$. Thus f is invertible with
$f^{-1} = f$. Therefore f is a bijection

$$\therefore \quad |S^{(k)}| = |S^{(n-k)}|$$

$$\therefore \quad \binom{n}{k} = \binom{n}{n-k}$$

$\blacksquare$