CSE 16 5-16-24

- Quiz 3 today: 4:25 – 4:55

6.2 Pigeonhole Principle

Theorem

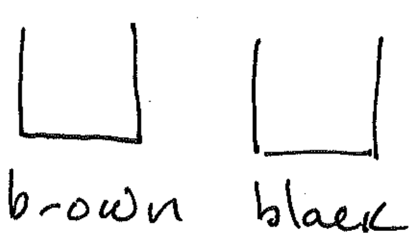
If $k+1$ (or more) objects are placed in k boxes, then (at least) 1 box contains (at least) 2 objects.

Ex.

A drawer contains 12 black socks & 12 brown socks. A man removes socks in the dark.

- What is the smallest number of socks whose removal would guarantee 2 of same color.

Pigeons ~ socks
Pigeonholes ~ colors



answer: 3

<u>Aside</u>: floor $\lfloor \cdot \rfloor$, ceiling $\lceil \, \rceil$

floor $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$

ceiling $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$

- $\lfloor x \rfloor$ = greatest int. l.t.e.t x

  equivalently:

  $$n = \lfloor x \rfloor \quad \text{iff} \quad n \le x < n+1$$

- $\lceil x \rceil$ = least int. g.t.e.t.

  equivalently

  $$n = \lceil x \rceil \quad \text{iff} \quad n-1 < x \le n$$

<u>also</u>: $\lfloor x \rfloor$ and $\lceil x \rceil$ are unique ints. satisfying:

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$$

Theorem (Generalized. P.P.)

If n (or more) objects are placed in k boxes, then (at least) 1 box contains (at least) $\lceil \frac{n}{k} \rceil$ objects.

note: $\lceil \frac{k+1}{k} \rceil = \lceil 1 + \frac{1}{k} \rceil = 2$

Proof (contradiction)

Assume each box contains at most $(\lceil \frac{n}{k} \rceil - 1)$ objects. Then the # of objects n satisfies

$$n \le k \cdot (\lceil \tfrac{n}{k} \rceil - 1) < k\left((\tfrac{n}{k} + \cancel{1}) - \cancel{1}\right) = \cancel{k} \cdot \frac{n}{\cancel{k}} = n$$

$\therefore n < n$. X .

$\blacksquare$

Ex.

What is the smallest # of socks whose removal guarantees 6 of the same color.

let n be the answer.

objects : socks n

boxes : colors 2

we seek smallest n s.t.

$$\left\lceil \frac{n}{2} \right\rceil = 6$$

$$\therefore \quad 5 < \frac{n}{2} \leq 6$$

$$\therefore \quad \boxed{10 < n} \leq 12$$

$$\therefore \quad \boxed{n = 11}$$

Ex.

How many students must be enrolled to guarantee at least 100 were born in the same month?

objects: students n

boxes: months 12

find smallest n s.t.

$$\left\lceil \frac{n}{12} \right\rceil = 100$$

$$\therefore \quad 99 < \frac{n}{12} \leq 100$$

$$\therefore \quad \underbrace{12 \cdot 99 < n} \leq 12 \cdot 100$$

$$\therefore \quad n = 12 \cdot 99 + 1 = \boxed{1189}$$

Ex.

There are 38 different periods during which classes can be scheduled. If there are 677 different classes, how many rooms are needed?

objects : classes 677

boxes : periods 38

By the G.P.P. at least one period has

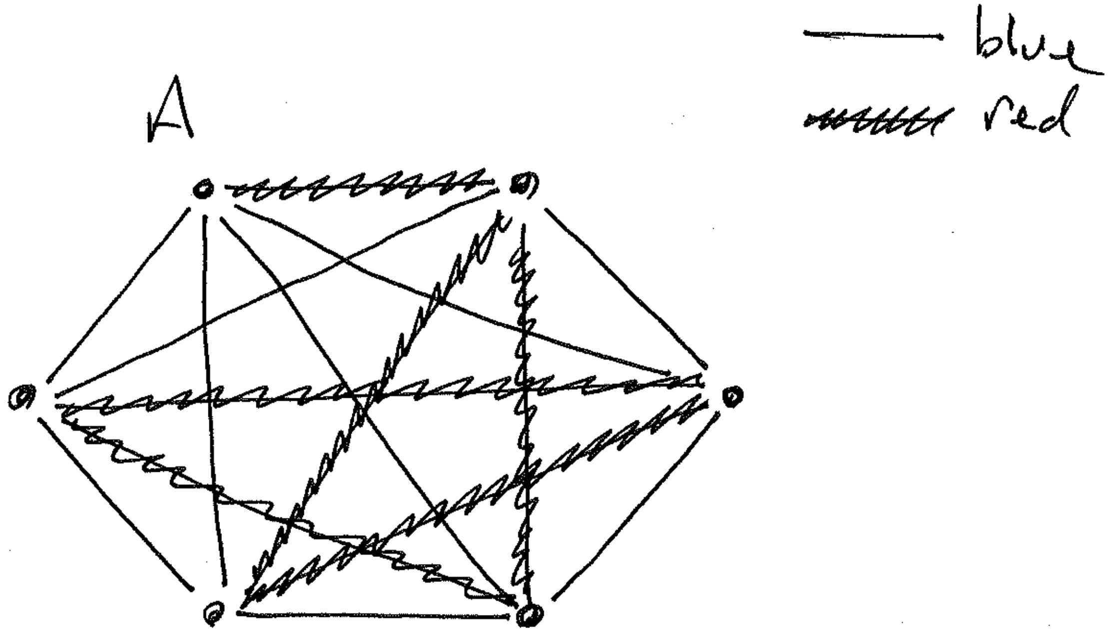
$$\left\lceil \frac{677}{38} \right\rceil = \lceil 17.8 \rceil = 18$$

classes. ∴ at least 18 rooms are necessary.

Theorem

In any group of 6 persons in which any 2 are either friends or enemies, there must exist either 3 mutual friends or 3 mutual enemies.

Picture



Proof

Let A be a person in the Group. The remaining 5 fall into 2 classes

$$F = \{\text{friends of } A\}$$

$$E = \{\text{enemies of } A\}$$

By the GPP at least $\lceil \frac{5}{2} \rceil = 3$ People are in same class.

Case 1: $|F| \geq 3$

Say B, C, D are friends of A.

If any 2 of these are friends (of each other), say B, C, then $\{A, B, C\}$ is a group of 3 mutual friends.

If no two of these are friends of each other, then $\{B, C, D\}$ is a group of 3 mutual enemies.

<u>Case 2</u>: $|E| \geq 3$.

Same argument but swap

friend $\longleftrightarrow$ enemy

$\blacksquare$