CSE 16 5-14-24

(5.3) ....

Exercise

Prove that

$$\forall n \geq 1 : GCD(F_n, F_{n+1}) = 1$$

where $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$

are the Fibonacci numbers.

(6.1) Counting (combinatorics)

Defn

A *string* is a finite sequence of symbols from some alphabet, S.

usually, $|S| < \infty$.

Ex.

how many strings of len. 2 are there from $S = \{a, b, c, \ldots, x, y, z\}$?

$$\begin{array}{cc} & \overbrace{\hphantom{aa \quad ab \quad ac \quad \cdots \quad az}}^{26} \\ 26 & \left[\begin{array}{ccccc} aa & ab & ac & \cdots & az \\ ba & bb & bc & \cdots & bz \\ \vdots & \vdots & \vdots & & \vdots \\ za & zb & zc & \cdots & zz \end{array}\right. \end{array}$$

answer $= 26 \cdot 26 = 26^2 = \boxed{676}$

Ex.

how many such strings of len 5 ?

$$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$

$$26 \ 26 \ 26 \ 26 \ 26 = 26^5 = \boxed{11,881,376}$$

Ex. how many such strings have no repeated letters ?

#choices	26	25	24	23	22
Positions :	1	2	3	4	5

$$\text{ans} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \boxed{7,893,600}$$

Ex.

how many bit strings of len k are there? alphabet $= \{0, 1\}$

#choice

$$\underset{1}{\underline{2}} \quad \underset{2}{\underline{2}} \quad \underset{3}{\underline{2}} \quad \cdots\cdots \quad \underset{k}{\underline{2}}$$

ans. $= 2^k$.

Ex. how many strings of len. k from an alphabet of size n?

$$\boxed{\#\text{strings} = n^k}$$

Ex. how many such strings have no repeated letters?

#choices:	n	$(n-1)$	$(n-2)$		$(n-k+1)$
Position:	1	2	3	....	k

$$\boxed{\begin{aligned}(\#\text{strings w/o rep.}) &= n\cdot(n-1)(n-2)\cdots(n-k+1)\\ &= \frac{n!}{(n-k)!}\end{aligned}}$$

In general:

$$(\#\text{strs w.o. rep.}) = \begin{cases} \dfrac{n!}{(n-k)!} & (0 \le k \le n) \\ 0 & k > n \end{cases}$$

Product Rule

Suppose a task T is decomposed into subtasks $T_1, T_2, \ldots, T_k$ in such a way that after

$$T_1, T_2, \ldots, T_{i-1}$$

have been performed, there are n_i ways to perform T_i. Then T can be performed in

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

ways.

<u>Theorem</u>

If A, B are finite sets, then so is $A \times B$ and

$$|A \times B| = |A| \cdot |B| .$$

<u>Theorem</u>

If S is a finite set, so is $\mathbb{P}(S)$ and

$$|\mathbb{P}(S)| = 2^{|S|}$$

Proof.

task T is to construct a subset

$$A \subseteq S.$$

let $S = \{x_1, x_2, \ldots, x_n\}$. Then

T decomposes as:

	choose	# ways
T_1 :	$x_1 \in A$ or $x_1 \notin A$	2
T_2 :	$x_2 \in A$ or $x_2 \in A$	2
$\vdots$	$\vdots$	$\vdots$
T_n :	$x_n \in A$ or $x_n \in A$	2

Therefore

$$|\mathbb{P}(S)| = 2 \cdot 2 \ldots 2 = 2^n = 2^{|S|}.$$

∎

Defn

Let A, B be sets. The set of functions with domain A and codomain B is denoted

$$B^A = \{f : A \to B\}$$

Theorem

If A, B are finite sets, then so is B^A and

$$|B^A| = |B|^{|A|}$$

Proof

Task T is to construct a function $f : A \to B$. Suppose $|A| = k$, $|B| = n$, and

$$A = \{x_1, x_2, \ldots, x_k\}$$

decompose T into

	choose	# ways
T_1 :	$f(x_1) \in B$	n
T_2 :	$f(x_2) \in B$	n
⋮		
T_k :	$f(x_k) \in B$	n

$$\therefore \; |B^A| = n^k = |B|^{|A|}$$

∎

Exercise

show that the # of injective functions $f: A \to B$, $|A| = k$, $|B| = n$ is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Exercise

- find a bijection

$$\left\{\begin{array}{l}\text{strs of len } k \\ \text{from alpha size } n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{functions } f: A \to B \\ |A| = k,\ |B| = n\end{array}\right\}$$

- restrict this map

$$\left\{\begin{array}{l}\text{strs w.o.} \\ \text{repetition}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{injections} \\ f: A \to B\end{array}\right\}$$

Sum Rule

Suppose a task T can be performed by doing exactly one of the subtasks

$$T_1, T_2, \ldots, T_k$$

where each T_i can be performed in n_i ways $(1 \le i \le k)$. Then T can be performed in

$$n_1 + n_2 + \cdots + n_k$$

ways

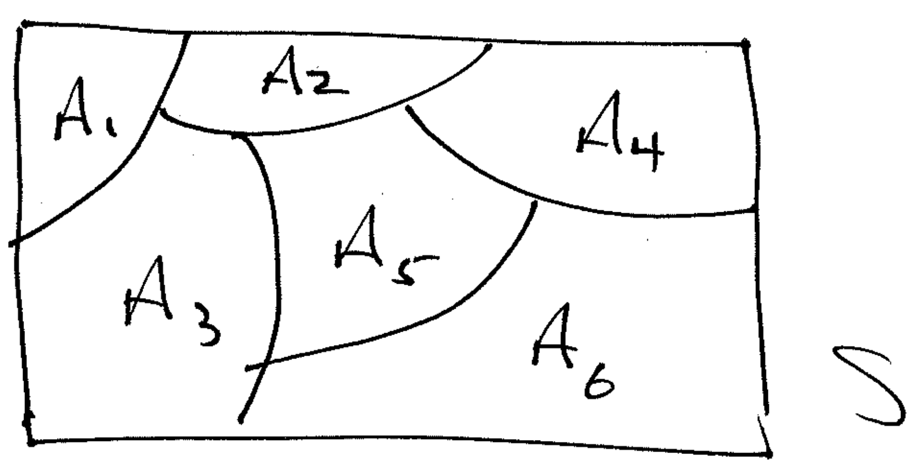
Theorem

Let $A_1, A_2, \ldots, A_k$ be finite sets that are pairwise disjoint.

$$A_i \cap A_j = \phi \quad (i \neq j)$$

Then $S = A_1 \cup A_2 \cup \cdots \cup A_k$ is also finite and

$$|S| = |A_1| + |A_2| + \cdots + |A_k| .$$



Ex.

how many strings of len. at most 5 can be formed from an alphabet of size 12?

	#ways
T_5 : choose str. of len 5 :	12^5
T_4 : " " " " 4 :	12^4
T_3 : -- -- -- 3 :	12^3
T_2 : -- -- -- 2 :	12^2
T_1 : -- -- -- 1 :	12^1
T_0 : -- -- -- 0 :	12^0

By sum rule

$$\#strs = 12^0 + 12^1 + \cdots + 12^5 = \frac{12^6 - 1}{12 - 1} = \boxed{271,453}$$

Principle of Inclusion-Exclusion

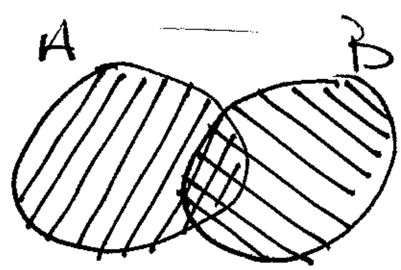
Special case of sum rule:

$$A \cap B = \phi$$

Then

$$|A \cup B| = |A| + |B|.$$

If $A \cap B \neq \phi$, then



$A \cup B$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex. how many bit strings of len. 7 either begin with '00' or end with '111'.

Let

	# strs
$A = \{00xxxxx\}$	2^5
$B = \{xxxx111\}$	2^4
$A \cap B = \{00xx111\}$	2^2

$$|A \cup B| = 2^5 + 2^4 - 2^2 = \boxed{44}$$

Defn

The Euler Totient Function

$\phi(n)$ is the number of numbers in $\{1, 2, \ldots, n\}$ that are relatively prime to n.

Ex.

Determine $\phi(1000)$

$$\phi(1000) = 1000 - (\#\text{ of }\#\text{s not R.P. to } 1000)$$

A number is not rel. prime to

$$1000 = 2^3 \cdot 5^3$$

iff it is divisible by 2 or divisible by 5, or by both.

$$(\#\text{ of } \#\text{s div by } 2) = \frac{1000}{2} = 500$$

$$(\#\text{ of } \#\text{s div by } 5) = \frac{1000}{5} = 200$$

$$(\#\text{ of } \#\text{s div by } 10) = \frac{1000}{10} = 100$$

Thus

$$\phi(1000) = 1000 - (500 + 200 - 100)$$

$$= 1000 - 600$$

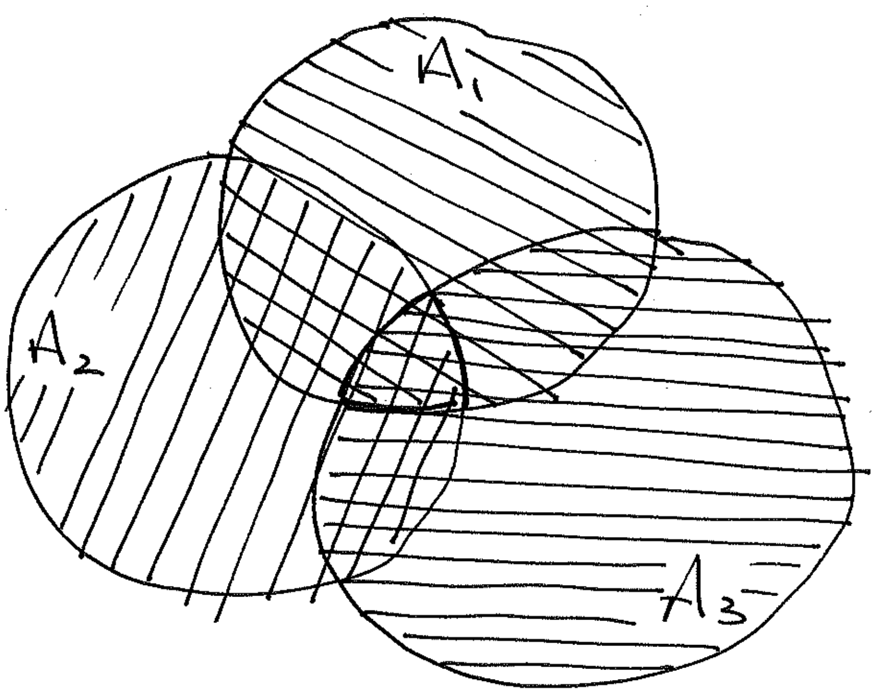
$$= \boxed{400}$$

Gene-ralizations of PIE

- $|A_1 \cup A_2 \cup A_3| = (|A_1| + |A_2| + |A_3|)$

  $- (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)$

  $+ |A_1 \cap A_2 \cap A_3|$



- $$\left|\bigcup_{i=1}^{4} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

- $$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| \;\vdots\; + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Ex. let p, q be primes. Then

$$\phi(p^n q^m) = p^n q^m - \left(p^{n-1} q^m + p^n q^{m-1} - p^{n-1} q^{m-1}\right)$$

$$= \text{--- algebra} \ldots$$

$$= \left(p^n - p^{n-1}\right)\left(q^m - q^{m-1}\right)$$

$$= \phi(p^n) \cdot \phi(q^m)$$

Exercise find $\phi(p^n q^m r^k)$ where p, q, r are primes