CSE 16 5-10-24

Supplemental Lecture

Theorem

Every positive integer can be expressed as a product of (zero or more) primes.

Proof

Let $P(n) = \boxed{\text{'} n \text{ is a product of primes'}}$

I. $P(1)$ says that 1 is a product of primes. indeed, 1 is the empty product, so base case is proved.

II d. $\forall n > 1 : (P(1) \wedge \ldots \wedge P(n-1)) \rightarrow P(n)$

Let $n > 1$, and assume $P(1), P(2), \ldots, P(n-1)$ are all true, i.e. $P(k)$ is true for k in the range $1 \leq k < n$. In other words any k in range $1 \leq k < n$ can be written as a product of primes.

We must show $P(n)$ is true, i.e. n can be expressed as a product of primes.

If n is a prime, then we're done since n is a product of one prime.

Otherwise n is composite. Then

$$n = a \cdot b$$

where $1 < a < n$ and $1 < b < n$. By the induction hypothesis, both $P(a)$ and $P(b)$ are true, so

$$a = p_1 p_2 \cdots p_\ell$$

and

$$b = q_1 q_2 \cdots q_m$$

where all p_i, q_j $(1 \leq i \leq \ell, 1 \leq j \leq m)$ are prime. But then

$$n = a \cdot b = p_1 \cdots p_\ell \cdot q_1 \cdots q_m ,$$

hence n is a product of primes.

By the 2^{ND} PMI, every $n \in \mathbb{Z}^+$ is a product of primes. ∎

Multiple base cases

Ex. $\forall n \geq 2 : \boxed{\exists x \geq 0, \exists y \geq 0 : n = 2x + 3y}$ ← $P(n)$

Proof

I. two base cases

$P(2) : 2 = 2 \cdot 1 + 3 \cdot 0$ ✓

$P(3) : 3 = 2 \cdot 0 + 3 \cdot 1$ ✓

IId. $\forall n > 3 : (P(2) \wedge \ldots \wedge P(n-1)) \rightarrow P(n)$

Let $n > 3$, i.e. $n \geq 4$. Assume for all k in the range $2 \leq k < n$ that

$$\exists x' \geq 0, \exists y' \geq 0 : k = 2x' + 3y'.$$

we must show

$$\exists x \geq 0, \exists y \geq 0 : n = 2x + 3y$$

Since $n \geq 4$ we have $2 \leq n-2 < n$, so by the induction hypothesis, we have $x' \geq 0$ and $y' \geq 0$ such that

$$n-2 = 2x' + 3y'$$

Let $x = x'+1$ and $y = y'$. Then $x \geq 0$, $y \geq 0$, and

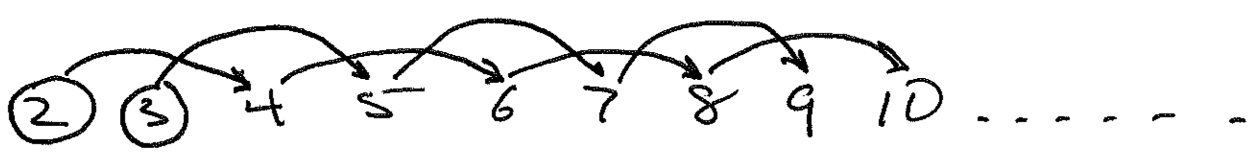
$$\begin{aligned} n &= (n-2)+2 \\ &= (2x'+3y')+2 \\ &= 2(x'+1)+3y' \\ &= 2x+3y \end{aligned}$$

as required. $\blacksquare$

Rmk

- used strong ind. since n^{th} domino was not toppled by $(n-1)^{th}$.

- why 2 base cases? same reason.



Ex. same but now weak ind, and one base case.

$$\forall n \geq 2 \ \boxed{\exists x \geq 0, \exists y \geq 0 : n = 2x + 3y}$$

Proof (weak form II b)

I. $P(2)$: $2 = 2 \cdot 1 + 3 \cdot 0$. ✓

II b $\forall n > 2 : P(n-1) \rightarrow P(n)$

Let $n > 2$. Assume $\exists x' \geq 0, \exists y' \geq 0$ s.t.

$$n - 1 = 2x' + 3y'.$$

we must show $\exists x \geq 0, \exists y \geq 0$ s.t.

$$n = 2x + 3y$$

Since $n > 2 \rightarrow n-1 \geq 2 > 0$, we have that not both $x' = 0$ and $y' = 0$

<u>case 1</u>: $x' > 0$

Let $x = x' - 1$ and $y = y' + 1$. Then $x \geq 0$ and $y \geq 0$ and

$$n = (n-1)+1$$
$$= (2x'+3y')+1 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$
$$= 2x' + 3y' + 3 - 2$$
$$= 2(x'-1) + 3(y'+1)$$
$$= 2x + 3y$$

<u>Case 2</u> $y' > 0$

Let $x = x'+2$, $y = y'-1$. Then

$x \geq 0$, $y \geq 0$, and

$$n = (n-1)+1$$
$$= (2x'+3y')+1 \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$
$$= 2x' + 3y' + 4 - 3$$
$$= (2x'+4) + (3y'-3)$$
$$= 2(x'+2) + 3(y'-1)$$
$$= 2x + 3y.$$

In either case there exist $x \geq 0, y \geq 0$ s.t. $n = 2x + 3y$, we have the result for all $n \geq 2$, By the 1st PMI.

5.3 Recursive Definitions

Recall the Fibonacci Sequence

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} & (n \geq 2) \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	...
F_n	0	1	1	2	3	5	8	13	21	34	...

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} F_{2k-1} = F_{2n}$

Proof.

I. if $n=1$ then we have

$$F_{2\cdot 1-1} = F_{2\cdot 1}$$

i.e. $F_1 = F_2$ ✓

IIa. $\forall n \geq 1 : P(n) \to P(n+1)$

let $n \geq 1$. assume

$$\sum_{k=1}^{n} F_{2k-1} = F_{2n}.$$

must show

$$\sum_{k=1}^{n+1} F_{2k-1} = F_{2n+2}.$$

Then

$$\sum_{k=1}^{n+1} F_{2k-1} = \left(\sum_{k=1}^{n} F_{2k-1}\right) + F_{2(n+1)-1}$$

$$= F_{2n} + F_{2n+1} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{ind. hyp.} \end{array} \right.$$

$$= F_{2n+2} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{Fib. rec.} \end{array} \right.$$

■

<u>Ex.</u> $\forall n \geq 1: \sum_{k=1}^{n} F_{k-1} \cdot F_k = \begin{cases} F_n^{\,2} & (n \text{ even}) \\ F_{n-1} \cdot F_{n+1} & (n \text{ odd}) \end{cases}$

Proof

I. if $n=1$, then we have

$$F_0 \cdot F_1 = F_0 \cdot F_2$$

i.e. $0 = 0$ ✓

IIa. $\forall n \geq 1$: $P(n) \rightarrow P(n+1)$.

Let $n \geq 1$. Assume

$$\sum_{k=1}^{n} F_{k-1} F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$

must show

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \begin{cases} F_{n+1}^2 & (n+1 \text{ even}) \\ F_n F_{n+2} & (n+1 \text{ odd}) \end{cases}$$

Then

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \left(\sum_{k=1}^{n} F_{k-1} F_k \right) + F_n F_{n+1}$$

$$= \left\{ \begin{array}{ll} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{array} \right\} + F_n F_{n+1} \quad \boxed{\text{by ind. hyp.}}$$

$$= \begin{cases} F_n^2 + F_n F_{n+1} & (n \text{ even}) \\ F_{n-1} F_{n+1} + F_n F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_n (F_n + F_{n+1}) & (n \text{ even}) \\ (F_{n-1} + F_n) F_{n+1} & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_n \cdot F_{n+2} & (n \text{ even}) \\ F_{n+1}^2 & (n \text{ odd}) \end{cases}$$

$$= \begin{cases} F_n F_{n+2} & (n+1 \text{ odd}) \\ F_{n+1}^2 & (n+1 \text{ even}) \end{cases}$$

$$= \begin{cases} F_{n+1}^2 & (n+1 \text{ even}) \\ F_n F_{n+2} & (n+1 \text{ odd}) \end{cases}$$

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