cse 16 4-9-24

Let $P(x,y)$, $Q(z)$ be Predicates

Then

$$R(x,y,z) = P(x,y) \vee Q(z)$$

$$S(y,z) = \forall x \, P(x,y) \vee Q(z)$$

$$T(y) = \exists z \, \forall x \big(P(x,y) \vee Q(z) \big)$$

free

bound bound

Translation

Ex. 'all horses in California are blue'

- let $U = \{$horses in California$\}$

  $B(x) =$ 'x is Blue'.

$$\forall x \, B(x)$$

- let $U = \{$horses$\}$

  $C(x) =$ 'x lives in California'

$$\forall x \, (C(x) \rightarrow B(x))$$

wrong: $\forall x \, (C(x) \wedge B(x))$

- $U = \{\text{living things}\}$

  $H(x) =$ 'x is a horse'

$$\forall x \left((H(x) \wedge C(x)) \rightarrow B(x) \right)$$

other suggestions:

$$\forall x \left((H(x) \rightarrow C(x)) \rightarrow B(x) \right)$$

Question $\quad p \wedge q \rightarrow r \overset{?}{\equiv} (p \rightarrow q) \rightarrow r$

1.5 Nested Quantifiers

What is meaning of '$\equiv$' logical equivalence for quantified expressions?

Defn

Let A, B be expressions containing Predicates & Quantifiers, and in which all variables are bound. Then

$$A \equiv B$$

iff A and B have the same truth value for all Predicates,

<u>Ex.</u> $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

<u>Ex.</u> $\neg \forall x\, P(x) \not\equiv \forall x\, \neg P(x)$

why? let $P(x) = $ '$x > 0$', $U = \mathbb{R}$.

Then

L.H.S. is True

R.H.S. is False

Let $P(x,y)$ be a Prop. fcn. with $\mathcal{U}_x = \mathcal{U}_y = \mathcal{U}$. Then

(1) $\forall_x \forall_y P(x,y) \equiv \forall_y \forall_x P(x,y)$

(2) $\exists_x \exists_y P(x,y) \equiv \exists_y \exists_x P(x,y)$

(3) $\forall_x \exists_y P(x,y) \not\equiv \exists_y \forall_x P(x,y)$

<u>Exercise</u>: illustrate (1) and (2) on $\mathcal{U} = \{1, 2\}$.

why is (3) $\not\equiv$?

let $P(x,y) = 'x<y'$, $\mathcal{U} = \mathbb{R}$.

LHS: $\forall x \exists y : x < y$ True

RHS: $\exists y \forall x : x < y$ False

More Translations

$L(x,y) =$ 'x loves y', $\mathcal{U} = \{\text{People}\}$

(1) 'everybody loves somebody'

$$\forall x \, \exists y \; L(x,y)$$

(2) 'somebody love everybody'

$$\exists x \, \forall y \; L(x,y)$$

(3) 'nobody loves everybody'

$$\neg \exists x \forall y \, L(x,y) \equiv \forall x \, \neg \forall y \, L(x,y)$$

$$\equiv \forall x \, \exists y \, \neg L(x,y)$$

(4) 'there is exactly one person whom everyone loves'

$$\exists y \Big(\forall x\, L(x,y) \wedge \forall z \big((z \neq y) \rightarrow \exists w\, \neg L(w,z) \big) \Big)$$

(5) 'there is a person who loves everyone except himself'

$$\exists x \Big(\neg L(x,x) \wedge \forall y \big((y \neq x) \rightarrow L(x,y) \big) \Big)$$

Ex. let $U = \{\text{People}\}$

$E(x) =$ 'x is the King of England'

$F(x) =$ 'x is the King of France'

'no King of England is not the King of France'

$$\neg \exists x \, (E(x) \wedge \neg F(x))$$

$$\equiv \forall x \, \neg (E(x) \wedge \neg F(x))$$

$$\equiv \forall x \, (\neg E(x) \vee F(x))$$

$$\equiv \forall x \, (E(x) \rightarrow F(x))$$

Scope of Quantifiers

(1) $\forall x\,(P(x) \to Q(x)) \not\equiv (\forall x\, P(x)) \to (\forall x\, Q(x))$

(2) $\exists x\,(P(x) \to Q(x)) \not\equiv \exists x\, P(x) \to \exists x\, Q(x)$

(3) $\exists x\,(P(x) \land Q(x)) \not\equiv \exists x\, P(x) \land \exists x\, Q(x)$

- (1) and (2) are $\not\equiv$ since

  $U = \{\text{people}\}$

  $P(x)$ = 'x has green eyes'

  $Q(x)$ = 'x is 50 feet tall'

- exercise: prove (3).

<u>Exercise</u>: investigate

(4) $\forall x\,(P(x) \vee Q(x))$? $\forall x\,P(x) \vee \forall x\,Q(x)$

(5) $\exists x\,(P(x) \vee Q(x))$? $\exists x\,P(x) \vee \exists x\,Q(x)$

(6) $\forall x\,(P(x) \wedge Q(x))$? $\forall x\,P(x) \wedge \forall x\,Q(x)$

answers: (4) $\not\equiv$

(5) $\equiv$

(6) $\equiv$

1.7 Intro to Proofs

- we say $n \in \mathbb{Z}$ is <u>even</u>

$\uparrow$ the set of integers

iff there exists $k \in \mathbb{Z}$ such that $n = 2k$, i.e.

$\exists k : n = 2k$ (n is free)

- we say $n \in \mathbb{Z}$ is <u>odd</u> iff there exists $k \in \mathbb{Z}$ s.t. $n = 2k+1$

$\exists k : n = 2k+1$ (n is free)

<u>note</u>: every $n \in \mathbb{Z}$ is either even or odd. Proof later.