cse 16 4-4-24

1.3 Equivalences

Defn

a compound prop. that is true for every assignment of truth values to its propositional variables is called a tautology. A contradiction is a compound prop. that is false for every assignment.

Ex.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

Tautology contradiction

Ex. $p \rightarrow (p \vee q)$ is a tautology

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

A proposition that is not a contradiction is called <u>satisfiable</u>.

A <u>contingency</u> is a prop. that is neither a tautology nor a contradiction.

<u>Ex.</u> p

<u>Defn</u>

Two propositions A, B are said ~~to~~ to be <u>logically equivallent</u> iff

$$A \Leftrightarrow B$$

is a tautology.

notation: $A \equiv B$

or $A \Leftrightarrow B$

i.e. $A \equiv B$ iff A & B have the same truth table.

not necessary.

Ex. $p \to q \equiv (\neg p) \vee q$ ✓

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

tautology

Ex. $p \to q \equiv \neg q \to \neg p$ (contrapositive)

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Ex. $p \to q \not\equiv q \to p$

exercise.

Exercise

Prove all equivalences on tables

6, 7, 8 in 1.3

!!!
...

Table 7

- $\boxed{p \to q \equiv \neg p \vee q}$ #1
- $p \to q \equiv \neg q \to p$
- $p \vee q \equiv \neg p \to q$
- $p \wedge q \equiv \neg(p \to \neg q)$

⋮

Algebraic manipulations

<u>Ex.</u> show $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

	<u>reason</u>
$(p \vee q) \to r \equiv \neg(p \vee q) \vee r$	T.7 #1
$\equiv (\neg p \wedge \neg q) \vee r$	DeMorgan
$\equiv r \vee (\neg p \wedge \neg q)$	commotative
$\equiv (r \vee \neg p) \wedge (r \vee \neg q)$	distributive
$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$	commutative
$\equiv (p \to r) \wedge (q \to r)$	T.7 #1

$\boxed{}$

DeMorgan law:

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg(p \wedge q) = \neg p \vee \neg q$

Distributive

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Associative

- $p \vee (q \vee r) \equiv (p \vee q) \vee r$
- $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Ex. show $(p \wedge q) \rightarrow (p \rightarrow q)$ is a tautology

$$(p \wedge q) \rightarrow (p \rightarrow q) \equiv \neg(p \wedge q) \vee (\neg p \vee q) \quad \text{T.7 \#1}$$

$$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q) \quad \text{DeMorgan}$$

$$\equiv (\neg p \vee \neg p) \vee (\neg q \vee q) \quad \text{comm. \& Assoc.}$$

$$\equiv \neg p \vee T \quad \text{Idem. \& Negation}$$

$$\equiv T \quad \text{Domination}$$

■

Exercise: show

$$\neg(p \rightarrow q) \rightarrow p$$

is a tautology.

1.4 Predicates & Quantifiers

Defn

A Propositional function (or Predicate)

is a statement $P(x)$ that

becomes a Proposition when a

value of x is specified.

Ex. $P(x) =$ '$x < 7$' : $U = \mathbb{R}$

$P(2)$ is '$2 < 7$'

$P(10)$ is '$10 < 7$'

Ex. $P(x) =$ 'x is a sunny day'

$$U = \{days\}$$

Ex. $Q(x,y) =$ '$x + y = 10$'

$$U_x = U_y = \mathbb{R}$$

The Universe of Discourse

(or Domain) of a Prop. Function.

is the set of possible values

for its variable(s)

notation: U

Quantification

Defn

The *Universal Quantification* of $P(x)$ is the Proposition

'$P(x)$ is true for all $x \in \mathcal{U}$'

Notation: $\forall x \, P(x)$

or $\forall x \in \mathcal{U} : P(x)$

Read: 'for all x, $P(x)$'

<u>Defn</u>

The <u>Existential Quantification</u>

of $P(x)$ is the proposition

'$P(x)$ is true for at least one $x \in \mathcal{U}$'

<u>Notation</u>: $\exists x \; P(x)$

or $\exists x \in \mathcal{U} : P(x)$

<u>Read</u>: 'there exists x such that $P(x)$'.

or 'for some x, $P(x)$'

Ex $P(x) =$ '$x^2 \geq 0$' , $U = \mathbb{R}$

$\forall x\, P(x) =$ 'the square of any real number is non-negative'

$\exists x\, P(x) =$ 'the square of at least one real number is non-negative'

Ex. $Q(x) =$ '$x = 50$', $U = \mathbb{R}$

$\forall x\, Q(x) =$ 'every real number is 50'

$\exists x\, Q(x) =$ 'some real no. is 50'

Ex. $R(x) =$ '$2x = 3$', $U = \mathbb{R}$

$\forall x\, R(x)$ is false

$\exists x\, R(x)$ is true

Consider $U = \{1, 2\}$. Then

$$\forall x\, P(x) \equiv P(1) \wedge P(2)$$

$$\exists x\, P(x) \equiv P(1) \vee P(2)$$

so

$$\neg \forall x\, P(x) \equiv \neg \left(P(1) \wedge P(2)\right)$$

$$\equiv \neg P(1) \vee \neg P(2)$$

$$\equiv \exists x\, \neg P(x)$$

<u>Law</u>:

$$\boxed{\neg \forall x\, P(x) = \exists x\, \neg P(x)}$$

and

$$\neg \exists x P(x) \equiv \neg (P(1) \vee P(2))$$

$$\equiv \neg P(1) \wedge \neg P(2)$$

$$\equiv \forall x \neg P(x)$$

law: $\boxed{\neg \exists x P(x) = \forall x \neg P(x)}$

These hold for <u>all</u> universes.

Summary

	True when	False when
$\forall x\, P(x)$	$P(x)$ is true for all $x \in U$	$P(x)$ is False for at least one $x \in U$
$\exists x\, P(x)$	$P(x)$ is true for at least one $x \in U$	$P(x)$ is False for all $x \in U$

A variable x is called

- Bound: if it is either quantified or substituted
- Free: if it is not Bound

Ex.

	x	y	z
$\neg P(x,y) \vee Q(z)$	f	f	f
$\forall x\, \neg P(x,y) \vee Q(z)$	b	f	f
$\exists x\, \forall z\, \neg P(x,y) \vee Q(z)$	b	Y	b
$\exists z\, \neg P(6,8) \vee Q(z)$	b	b	b